

Contents

Foreword	
Introduction	1
0.1 What Is This Book About?	1
0.2 Now, What Is This Book <i>Really</i> About?	2
0.3 What Is This Book <i>Not</i> About?	4
0.4 Are There any Applications of This Theory?	5
0.5 How Was This Book Written?	5
0.6 What Is a Tutorial?	7
0.7 What Is CoCoA?	7
0.8 And What Is This Book Good for?	8
0.9 Some Final Words of Wisdom	9
4. The Homogeneous Case	11
4.1 Polynomial Rings Graded by Matrices	15
4.1.A Gradings by Matrices	16
4.1.B Graded Modules	20
4.1.C Gradings of Positive Type	23
<i>Tutorial 45.</i> Homogeneous Maps and Toric Ideals	29
<i>Tutorial 46.</i> Projective Varieties	30
4.2 Degree Forms and Macaulay Bases	32
4.2.A Positive Gradings	33
4.2.B Macaulay Bases	36
<i>Tutorial 47.</i> Computation of Macaulay Bases	40
<i>Tutorial 48.</i> Characterizations of Macaulay Bases	42
4.3 Homogenization	44
4.3.A Homogenization of Polynomials and Ideals	45
4.3.B Macaulay Bases and Homogenization	55
<i>Tutorial 49.</i> Homogenization of Modules	60
<i>Tutorial 50.</i> The Homogeneous Part of an Ideal	63
<i>Tutorial 51.</i> Implicitization and Homogenization	64
<i>Tutorial 52.</i> Projective Closure	66
4.4 Term Orderings of DegRev Type	68
<i>Tutorial 53.</i> Reduced Gröbner Bases and Homogenization	78

	<i>Tutorial 54.</i> Regular Sequences of Indeterminates	80
	<i>Tutorial 55.</i> Set-Theoretic Complete Intersections	82
4.5	Homogeneous Gröbner Bases	85
4.5.A	The Homogeneous Buchberger Algorithm	87
4.5.B	Truncated Gröbner Bases	92
	<i>Tutorial 56.</i> Computation of Homogeneous Gröbner Bases	95
	<i>Tutorial 57.</i> Some Applications of Truncated Gröbner Bases	98
4.6	Minimal Homogeneous Systems of Generators	100
	<i>Tutorial 58.</i> Computing Some Minimal Systems of Generators	110
	<i>Tutorial 59.</i> Optimizing the Homogeneous Buchberger Algorithm	111
4.7	Minimal Homogeneous Presentations	116
4.7.A	Existence and Uniqueness of Minimal Presentations	117
4.7.B	Idealization of Graded Modules and Presentations	123
4.7.C	Computation of Minimal Homogeneous Presentations	130
	<i>Tutorial 60.</i> Computing Some Idealizations	143
	<i>Tutorial 61.</i> Computing Some Minimal Presentations	144
4.8	Minimal Graded Free Resolutions	147
4.8.A	Existence and Uniqueness of Minimal Free Resolutions	148
4.8.B	Computation of Minimal Graded Free Resolutions	154
	<i>Tutorial 62.</i> The Hilbert-Burch Theorem	167
	<i>Tutorial 63.</i> Computing Some Graded Betti Numbers	169
5.	Hilbert Functions	173
5.1	Basic Properties of Hilbert Functions	179
5.1.A	Integer Functions of Polynomial Type	180
5.1.B	Hilbert Functions in the Standard Graded Case	185
	<i>Tutorial 64.</i> Hilbert Functions and Graded Free Resolutions	191
5.2	Hilbert Series	194
5.2.A	Univariate Power Series	195
5.2.B	Hilbert Series in the Standard Graded Case	202
	<i>Tutorial 65.</i> Knight Moves	205
	<i>Tutorial 66.</i> Veronese Subrings	209
	<i>Tutorial 67.</i> Powers of Polynomials and Ehrhart Functions	210
5.3	Computation of Hilbert Series	214
	<i>Tutorial 68.</i> Implementation of the Hilbert Series Algorithm	226
	<i>Tutorial 69.</i> Hilbert Driven Gröbner Basis Computations I	228
5.4	Dimension, Multiplicity, and Hilbert Polynomials	232
5.4.A	Dimension and Multiplicity of Standard Algebras	233
5.4.B	Hilbert Polynomials in the Standard Graded Case	239
	<i>Tutorial 70.</i> Computing the Dimension of a Module	243
	<i>Tutorial 71.</i> Chess Puzzles	244
	<i>Tutorial 72.</i> Photogrammetry	247
5.5	Bounds for Hilbert Functions	252
5.5.A	Binomial Representations	254
5.5.B	Lex-Segment Spaces and Ideals	258

5.5.C	The Theorems of Macaulay and Green	262
	<i>Tutorial 73.</i> Operations on Binomial Representations	271
	<i>Tutorial 74.</i> Bounds for Minimal Generators	273
	<i>Tutorial 75.</i> Gin for the Strongly Stable	274
5.6	Affine Hilbert Functions and Krull Dimension	278
5.6.A	The Hilbert Function of an Affine Algebra	279
5.6.B	Primary Decomposition in Noetherian Rings	284
5.6.C	The Krull Dimension of an Affine Algebra	291
	<i>Tutorial 76.</i> The Multiplicity of an Affine Algebra	297
	<i>Tutorial 77.</i> Primary Decomposition of Monomial Ideals	298
5.7	Independent Sets of Indeterminates	301
5.7.A	The Combinatorial Dimension of an Affine Algebra	301
5.7.B	Transcendence Degrees	308
	<i>Tutorial 78.</i> Noether Normalization	312
	<i>Tutorial 79.</i> Primary Decompositions II	315
5.8	General Hilbert Functions	321
5.8.A	Rings of Multivariate Laurent Series	323
5.8.B	Hilbert Functions in the General Case	325
5.8.C	Change of Gradings	332
	<i>Tutorial 80.</i> Hilbert Driven Gröbner Basis Computations II	335
	<i>Tutorial 81.</i> Rees Rings	339
	<i>Tutorial 82.</i> Segre Products and Hadamard Series	341
	<i>Tutorial 83.</i> A Toy Example	344
6.	Further Applications	347
6.1	Toric Ideals and Hilbert Bases	351
6.1.A	Toric Ideals	352
6.1.B	Hilbert Bases	358
	<i>Tutorial 84.</i> Magic Squares	367
	<i>Tutorial 85.</i> Computing the Gaps	370
	<i>Tutorial 86.</i> Matrices of Positive Type	372
6.2	Liftings of Ideals and Distractions	375
	<i>Tutorial 87.</i> SuperG Bases	384
6.3	Finite Sets of Points	387
6.3.A	Affine Point Sets	389
6.3.B	Projective Point Sets	395
6.3.C	Hilbert Functions of Points	404
	<i>Tutorial 88.</i> The Cayley-Bacharach Property	408
	<i>Tutorial 89.</i> Generic Sets of Points	411
	<i>Tutorial 90.</i> Error-Correcting Codes	415
6.4	Border Bases	419
6.4.A	Existence and Uniqueness of Border Bases	421
6.4.B	Characterizations of Border Bases	430
	<i>Tutorial 91.</i> Module Structures on Vector Spaces	442
	<i>Tutorial 92.</i> Design of Experiments	447

6.5	Filtrations	453
6.5.1	General Filtrations	454
6.5.B	Adic Filtrations and Tangent Cones	460
	<i>Tutorial 93.</i> Mora's Algorithm	465
	<i>Tutorial 94.</i> Hilbert Functions of Primary Ideals	470
	<i>Tutorial 95.</i> Singularities	472
6.6	SAGBI Bases	477
6.6.A	Definition and Basic Properties of SAGBI Bases	478
6.6.B	Characterization of SAGBI Bases	486
6.6.C	Computation of SAGBI Bases	494
	<i>Tutorial 96.</i> Variations on the SAGBI Theme	499
	<i>Tutorial 97.</i> Gröbner and SAGBI Bases Under Composition	503
	<i>Tutorial 98.</i> Molien's Theorem	507
6.7	Automatic Theorem Proving	509
6.7.A	The Tribulations of Automation	511
6.7.B	Algebraically True Statements	516
6.7.C	Optimal Hypothesis Ideals and Minimal Conditions ..	521
	<i>Tutorial 99.</i> To Prove or Not to Prove	528
A.	The ABC of CoCoA	535
B.	One Graphical Interface for Everybody	537
C.	More on CoCoA Programming	543
D.	Suggestions for Further Reading	557
E.	Hints for Selected Exercises	561
	Notation	565
	Bibliography	571
	Index	575