Multigraded Structures in Polynomial Rings

K field

$P = K[x_1, \ldots, x_n]$ polynomial ring

How can we equip P with a "good" grading?

• The set of degrees of "multihomogeneous" polynomials should be well-ordered.

• The homogeneous components of P should be finite-dimensional K-vector spaces.

• The graded version of Nakayama's Lemma should hold.

• There should be a term ordering which is compatible with the grading.

Positive Gradings.

P is graded by a matrix $W \in \mathrm{Mat}_{m,n}(\mathbb{Z})$

$\deg_W(x_i)$ is the i^{th} column of W

Example. $W = (1\ 1\ \cdots\ 1)$ defines the standard grading.

Definition. a) The grading given by W is called **weakly positive** if there is a linear combination of the rows of W which has positive entries only.

b) The grading given by W is called **positive** if the rank of W is m and if the first non-zero entry in each column of W is positive.

Proposition. a) Every positive grading is weakly positive.

b) If the grading given by W is weakly positive, then the homogeneous components of P and of finitely generated graded P-modules are finite dimensional K-vector spaces.

c) If the grading given by W is weakly positive, then the graded version of Nakayama's Lemma holds. In particular, there is a well-behaved notion of "minimal number of generators".

d) If the grading given by W is weakly positive, there is a (non-canonical) well-ordering on the set of degrees of homogeneous polynomials.

e) If the grading given by W is positive, then `Lex` is a well-ordering on the set of degrees of homogeneous polynomials.

f) If the grading given by W is positive, then there exists a degree compatible term ordering.

Situation.

$\delta_1, \ldots, \delta_r \in \mathbb{Z}^m$

$F = \bigoplus_{i=1}^{r} P(-\delta_i)$ graded free P-module

$M \subseteq F$ graded submodule

$\mathcal{V} = (v_1, \ldots, v_s)$ non-zero homogeneous generators of M

σ module term ordering on the terms in F

Whenever an element $g_i \in F$ occurs, we write

$$\mathrm{LM}_\sigma(g_i) = c_i\, t_i\, e_{\gamma_i}$$

where $c_i \in K$, t_i is a term, and $1 \leq \gamma_i \leq r$.

(i, j) such that $i < j$ and $\gamma_i = \gamma_j$ is called a **critical pair**

$t_{ij} = \frac{\mathrm{lcm}(t_i, t_j)}{t_i}$ for all i, j

$\sigma_{ij} = \frac{1}{c_i}\, t_{ij}\, e_i - \frac{1}{c_j}\, t_{ji}\, e_j$ **critical syzygy**

$S_{ij} = \frac{1}{c_i}\, t_{ij}\, g_i - \frac{1}{c_j}\, t_{ji}\, g_j$ **S-vector**

The Multihomogeneous Buchberger Algorithm

1) Let $B = \emptyset$, $\mathcal{W} = \mathcal{V}$, $\mathcal{G} = \emptyset$, and let $s' = 0$.

2) Let d be the smallest degree with respect to `Lex` of an element in B or in $\mathcal{W}$. Form the subset $B_d = \{(i,j) \in B \mid \deg_W(\sigma_{ij}) = d\}$ and the subtuple $\mathcal{W}_d$ of elements of degree d in $\mathcal{W}$, and delete their entries from B and $\mathcal{W}$, respectively.

3) If $B_d = \emptyset$, continue with step 6). Otherwise, choose a pair $(i,j) \in B_d$ and remove it from B_d.

4) Compute the S-vector S_{ij} and its normal remainder $S'_{ij} = \mathrm{NR}_{\sigma,\mathcal{G}}(S_{ij})$. If $S'_{ij} = 0$, continue with step 3).

5) Increase s' by one, append $g_{s'} = S'_{ij}$ to the tuple $\mathcal{G}$, and append $\{(i,s') \mid 1 \le i < s', \gamma_i = \gamma_{s'}\}$ to the set B. Continue with step 3).

6) If $\mathcal{W}_d = \emptyset$, continue with step 9). Otherwise, choose a vector $v \in \mathcal{W}_d$ and remove it from $\mathcal{W}_d$.

7) Compute $v' = \mathrm{NR}_{\sigma,\mathcal{G}}(v)$. If $v' = 0$, continue with step 6).

8) Increase s' by one, append $g_{s'} = v'$ to the tuple $\mathcal{G}$, and append $\{(i,s') \mid 1 \le i < s', \gamma_i = \gamma_{s'}\}$ to the set B. Continue with step 6).

9) If $B = \emptyset$ and $\mathcal{W} = \emptyset$, return the tuple $\mathcal{G}$ and stop. Otherwise, continue with step 2).

This is an algorithm which returns a deg-ordered tuple $\mathcal{G} = (g_1, \ldots, g_{s'})$ whose elements are a homogeneous σ-Gröbner basis of M.

The Buchberger Algorithm With Minimalization

1) Let $B = \emptyset$, $\mathcal{W} = \mathcal{V}$, $\mathcal{G} = \emptyset$, $s' = 0$, *and let* $\mathcal{V}_{\min} = \emptyset$.

2) Let d be the smallest degree with respect to `Lex` of an element in B or in $\mathcal{W}$. Form the subset $B_d = \{(i,j) \in B \mid \deg_W(\sigma_{ij}) = d\}$ and the subtuple $\mathcal{W}_d$ of elements of degree d in $\mathcal{W}$, and delete their entries from B and $\mathcal{W}$, respectively.

3) If $B_d = \emptyset$, continue with step 6). Otherwise, choose a pair $(i,j) \in B_d$ and remove it from B_d.

4) Compute the S-vector S_{ij} and its normal remainder $S'_{ij} = \mathrm{NR}_{\sigma,\mathcal{G}}(S_{ij})$. If $S'_{ij} = 0$, continue with step 3).

5) Increase s' by one, append $g_{s'} = S'_{ij}$ to the tuple $\mathcal{G}$, and append $\{(i,s') \mid 1 \le i < s', \gamma_i = \gamma_{s'}\}$ to the set B. Continue with step 3).

6) If $\mathcal{W}_d = \emptyset$, continue with step 9). Otherwise, choose a vector $v \in \mathcal{W}_d$ and remove it from $\mathcal{W}_d$.

7) Compute $v' = \mathrm{NR}_{\sigma,\mathcal{G}}(v)$. If $v' = 0$, continue with step 6).

8) Increase s' by one, append $g_{s'} = v'$ to $\mathcal{G}$ *and* v *to* $\mathcal{V}_{\min}$. Append $\{(i,s') \mid 1 \le i < s', \gamma_i = \gamma_{s'}\}$ to the set B. Continue with step 6).

9) If $B = \emptyset$ and $\mathcal{W} = \emptyset$, return $(\mathcal{G}, \mathcal{V}_{\min})$ and stop. Otherwise, continue with step 2).

This is an algorithm which returns a pair $(\mathcal{G}, \mathcal{V}_{\min})$ such that $\mathcal{G}$ is a deg-ordered tuple of homogeneous vectors which are a σ-Gröbner basis of M, and $\mathcal{V}_{\min}$ is a subtuple of $\mathcal{V}$ which is a minimal system of generators of M.

Idealization of Modules

Definition. Let R be a ring and M an R-module. We equip $R \times M$ with componentwise addition and the multiplication

$$(r, m) \cdot (r', m') = (r\, r',\, r\, m' + r'\, m)$$

In this way we get a ring $R \ltimes M$. We call it the **idealization** of M.

The canonical map $\imath(M) \longrightarrow R \ltimes M$ identifies M with an ideal $\imath(M)$ such that $\imath(M)^2 = 0$.

Idealization of a graded free module.

$F = \bigoplus_{i=1}^{r} P(-\delta_i)$ graded free P-module
$\overline{P} = K[x_1, \ldots, x_n,\, e_1, \ldots, e_r]$ polynomial ring
$\overline{W} = (W \mid \delta_1, \ldots, \delta_r)$ multigrading on $\overline{P}$
E ideal generated by $\{e_i\, e_j \mid 1 \le i \le j \le r\}$ in $\overline{P}$

Then the map $\varphi :\; P \ltimes F \;\longrightarrow\; \overline{P}/E$ defined by
$(f, (g_1, \ldots, g_r)) \;\longmapsto\; f + g_1 e_1 + \cdots + g_r e_r + E$
is an isomorphism of graded rings.

Idealization of a Graded Submodule.

$M \subseteq F$ graded submodule
$\mathcal{V} = (v_1, \dots, v_s)$ homogeneous system of generators of M
Under the map $M \hookrightarrow P \propto M \hookrightarrow P \propto F \xrightarrow{\varphi} \overline{P}/E$, the module M is identified with the residue class ideal of

$$I_M = (v_1, \dots, v_s) + E$$

Gröbner Bases and Idealization.

τ term ordering on $\mathbb{T}^n$
σ module term ordering on $\mathbb{T}^n \langle e_1, \dots, e_r \rangle$ which is compatible with τ
$\overline{\sigma}$ term ordering on $\mathbb{T}(x_1, \dots, x_n, e_1, \dots, e_r)$ which extends both τ and σ
G (reduced) σ-Gröbner basis of M

Then $G \cup \{e_i e_j \mid 1 \leq i \leq j \leq r\}$ is a (reduced) $\overline{\sigma}$-Gröbner basis of I_M.

Minimal Generators and Idealization.

Suppose that $M \subseteq (x_1, \ldots, x_n)F$.
$\mathcal{V} = (v_1, \ldots, v_s)$ minimal homogeneous system of generators of M

Then $\{v_1, \ldots, v_s\} \cup \{e_i e_j \mid 1 \le i \le j \le r\}$ is a minimal homogeneous system of generators of I_M.

Homogeneous Presentations.

$\eta_i = \deg_W(v_i)$ for $i = 1, \ldots, s$
$F' = \bigoplus_{i=1}^{s} P(-\eta_i)$ graded free P-module

$$F'' \xrightarrow{\ \psi\ } F' \xrightarrow{\ \varphi\ } M \longrightarrow 0$$

where $\varphi(\epsilon_i) = v_i$ for $i = 1, \ldots, s$, where F'' is a further graded free P-module, and where ψ is a homogeneous P-linear map.

If this sequence is exact, it is called a **homogeneous presentation** of M.

$\widetilde{P} = K[x_1, \ldots, x_n, e_1, \ldots, e_r, \epsilon_1, \ldots, \epsilon_s]$ polynomial ring
$\widetilde{W} = (W \mid \delta_1, \ldots, \delta_r \mid \eta_1, \ldots, \eta_s)$ defines a multigrading on $\widetilde{P}$
$\widetilde{E} = (e_i e_j)_{i,j} + (e_k \epsilon_\ell)_{k,\ell} + (\epsilon_\mu \epsilon_\nu)_{\mu,\nu}$ ideal in $\widetilde{P}$
If $(f_1, \ldots, f_s)$ is a syzygy of $(v_1, \ldots, v_s)$, then the corresponding element $f_1\epsilon_1 + \cdots + f_s\epsilon_s$ of $\widetilde{P}$ is contained in the ideal $(v_1 - \epsilon_1, \ldots, v_s - \epsilon_s)$.

Idealization of a Homogeneous Presentation.

The ideal $\tilde{I}_M = (v_1 - \epsilon_1, \ldots, v_s - \epsilon_s) + \widetilde{E}$ of $\widetilde{P}$ is called the **idealization of the presentation** of M given above.

a) There exists a unique P-algebra homomorphism

$$\Phi : P[\epsilon_1, \ldots, \epsilon_s]/(\epsilon_i \epsilon_j)_{i,j=1..s} \longrightarrow P[e_1, \ldots, e_r]/(e_i e_j)_{i,j=1..r}$$

which maps the residue class of ϵ_i to the residue class of v_i for $i = 1, \ldots, s$.

b) The image of Φ is the idealization of M.

c) Let I_M be the ideal in $P[e_1, \ldots, e_r]$ obtained by substituting $\epsilon_i \mapsto 0$ for $i = 1, \ldots, s$ in $\tilde{I}_M$. Then the idealization of M is the residue class ideal of I_M in the ring $P[e_1, \ldots, e_r]/(e_i e_j)_{i,j=1,\ldots,r}$.

d) The kernel of Φ is the idealization of $\mathrm{Syz}_P(\mathcal{V})$.

e) The idealization of $\mathrm{Syz}_P(\mathcal{V})$ corresponds to the residue class ideal of $\tilde{I}_M \cap P[\epsilon_1, \ldots, \epsilon_s]$ in $P[\epsilon_1, \ldots, \epsilon_s]/(\epsilon_i \epsilon_j)_{i,j=1..s}$.

The above homogeneous presentation of M is called **minimal** if it is of the form

$$F'' \xrightarrow{\mathcal{S}_{\min}} F' \xrightarrow{\mathcal{V}_{\min}} M \longrightarrow 0$$

where $\mathcal{V}_{\min}$ is a minimal homogeneous system of generators of M and where $\mathcal{S}_{\min}$ is a minimal homogeneous system of generators of $\mathrm{Syz}(\mathcal{V})$.

Computing Idealized Minimal Presentations

$M \subseteq F = \bigoplus_{i=1}^{r} P(-\delta_i)$, where $\delta_i >_{\mathrm{Lex}} 0$ for $i = 1, \ldots, r$

$\mathcal{V} = (v_1, \ldots, v_s)$ deg-ordered non-zero homogeneous generators of M such that $\eta_j = \deg_W(v_j) >_{\mathrm{Lex}} 0$ for $j = 1, \ldots, s$

$\widetilde{\sigma}$ elimination ordering for $\{e_1, \ldots, e_r\}$

1) Form the ideal $\tilde{I}_M = (v_1 - \epsilon_1, \ldots, v_s - \epsilon_s) + \widetilde{E}$ in $\widetilde{P}$.

2) Modify the Buchberger Algorithm with Minimalization such that it starts with $\mathcal{W} = (v_1 - \epsilon_1, \ldots, v_s - \epsilon_s)$, $\mathcal{G} = \mathcal{G}_{\min} = \mathcal{E}$, and $s' = r^2 - r + rs + s^2 - s$. Use it to compute a pair of tuples $(\mathcal{G}, \mathcal{G}_{\min})$ such that $\mathcal{G}$ is a homogeneous $\widetilde{\sigma}$-Gröbner basis of $\tilde{I}_M$ and $\mathcal{G}_{\min}$ is a homogeneous minimal system of generators of $\tilde{I}_M$.

3) Let $g_1, \ldots, g_\mu$ be the vectors in $\mathcal{G}_{\min}$ which are linear forms in $e_1, \ldots, e_r, \epsilon_1, \ldots, \epsilon_s$ and which contain at least one of the indeterminates $e_1, \ldots, e_r$. For $i = 1, \ldots, \mu$, let $\bar{g}_i$ be the polynomial obtained by substituting $\epsilon_j \mapsto 0$ for $j = 1, \ldots, s$ in g_i. Form the tuple $\overline{\mathcal{G}} = (\bar{g}_1, \ldots, \bar{g}_\mu)$.

4) Let $h_1, \ldots, h_\nu$ be the vectors in $\mathcal{G}_{\min}$ which are linear forms in $e_1, \ldots, e_r, \epsilon_1, \ldots, \epsilon_s$ and which contain none of the indeterminates $e_1, \ldots, e_r$. Form the tuple $\overline{\mathcal{S}} = (h_1, \ldots, h_\nu)$.

5) Return the pair $(\overline{\mathcal{G}}, \overline{\mathcal{S}})$ and stop.

This is an algorithm which returns a deg-ordered tuple $\overline{\mathcal{G}}$ of homogeneous vectors in F which generate M minimally and a deg-ordered tuple $\overline{\mathcal{S}}$ of homogeneous vectors in $F' = \bigoplus_{i=1}^{s} P(-\eta_i)$ which generate $\mathrm{Syz}_P(\mathcal{V})$ minimally.

Applications and Further Developments.

• One can apply to this algorithm all optimizations of the usual homogeneous Buchberger Algorithm, e.g. the optimized version described in
M. Caboara, M.K, and L. Robbiano, Efficiently computing minimal sets of critical pairs, preprint 2002.

• Using this description of the computation of the syzygy module, one can study **totally useless** critical pairs, i.e. critical pairs which reduce to zero during the Gröbner basis computation *and* which have the additional property that the syzygies they produce are not minimal generators of the syzygy module.

• One can extend the methods of this algorithm to compute the whole graded free resolution using one application of the multihomogeneous Buchberger algorithm *(horizontal strategy)*

The problem with this approach is that one does not know in advance how many additional indeterminates one needs for the higher syzygy modules.

• In order to further compactify the computation, M. Caboara and C. Traverso (work in progress) have succeeded to incorporate the whole computation in a polynomial ring $\overline{P} = K[x_1, \ldots, x_n, d, e, s]$ which requires only three additional indeterminates which roughly correspond to:

- s is the position of the module in the free resolution
- d is the multidegree of the corresponding standard basis vector
- e^i is the corresponding standard basis vector

This technique requires subtle extensions of the usual theory of module term orderings.