Multigraded Hilbert Series and Applications

by Martin Kreuzer (Dortmund)

and Lorenzo Robbiano (Genova)

Entertainment for Bored People

Find all triples of non-negative integers $(x, y, z) \in \mathbb{N}^3$ such that

$$3x - 5y + 4z = 0$$

1. Multigradings.

K field, $P = K[x_1, \ldots, x_n]$ polynomial ring

In this conference we shall encounter some bigradings (i.e. $\mathbb{Z}^2$-gradings) on P. Which conditions should a $\mathbb{Z}^m$-grading on P satisfy in order to be useful for computer algebra?

Wish 1: The set of degrees of homogeneous polynomials should be well-ordered. There should be no infinite sequence of stictly decreasing degrees. (Finiteness of algorithms!)

Wish 2: The graded version of Nakayama's Lemma should hold. There should be a well-behaved notion of "minimal number of generators".

Wish 3: The homogeneous components of finitely generated graded modules should be finite dimensional K-vector spaces. Hilbert functions and Hilbert series should exist.

Wish 4: A degree-compatible term ordering σ should exist. We should be able to compute homogeneous objects by using σ-Gröbner bases.

Wish 5: There should be an effective and well-defined way to compare degrees. One should be able to compute "degree-by-degree".

Definition. Let $w_1, \ldots, w_n \in \mathbb{Z}^m$.

a) There exists a unique $\mathbb{Z}^m$-grading on P such that $K \subseteq P_0$ and x_i is homogeneous of multidegree w_i for $i = 1, \ldots, n$. We form the matrix $W = (w_1\ w_2\ \cdots\ w_n)$ and say that P is **graded by** W.

b) The grading by W is called **weakly positive** if there exists a $\mathbb{Z}$-linear combination of the rows of W all of whose entries are positive.

c) The grading by W is called **positive** if $\text{rank}(W) = m$, no column of W is zero and the first non-zero entry in each column of W is positive.

Proposition. Let P be graded by $W \in \mathrm{Mat}_{m,n}(\mathbb{Z})$.

a) If the grading is weakly positive, then wishes 1), 2), and 3) come true.

b) If the grading is positive, then it is weakly positive. Moreover, wishes 1) – 5) come true.

The *proof* will be contained in
M. Kreuzer and L. Robbiano, Computational Commutative Algebra II, Springer Verlag 2004 (in preparation)

2. Multigraded Hilbert Series

From now on let P always be weakly positively graded by $W \in \mathrm{Mat}_{m,n}(\mathbb{Z})$.

Definition. Let M be a finitely generated graded P-module.

a) The map $\mathrm{HF}_M : \mathbb{Z}^m \longrightarrow \mathbb{Z}$ given by

$$\mathrm{HF}_M(i_1, \ldots, i_m) = \dim_K(M_{(i_1,\ldots,i_m)})$$

is called the **multigraded Hilbert function** of M.

b) The series $\mathrm{HS}_M(z_1, \ldots, z_m) = \sum_{i \in \mathbb{Z}^m} \mathrm{HF}_M(i)\, z_1^{i_1} \cdots z_m^{i_m}$ is called the **multigraded Hilbert series** of M.

Problem: Which object contains all multigraded Hilbert series? (It should be a ring!)

Remark. The group

$$\mathbb{Z}[[z_1, \ldots, z_m, z_1^{-1}, \ldots, z_m^{-1}]]$$

is not a ring, since $(1 + z_1 + z_1^2 + \cdots) \cdot (1 + z_1^{-1} + z_1^{-2} + \cdots)$ cannot be defined.

Example Let $P = K[x_1, x_2]$ be graded by $W = \binom{1\ \ 1}{0\ -1}$. Then the homogeneous component of multidegree (d_1, d_2) of P is $K \cdot x_1^{d_1+d_2} x_2^{-d_2}$. Therefore we have

$$\mathrm{HS}_P(z_1, z_2) = \textstyle\sum_{(d_1,d_2)\in\mathbb{Z}^2} z_1^{d_1+d_2} z_2^{-d_2}$$

Thus the multivariate Laurent series ring $\mathbb{Z}[[z_1, \ldots, z_m]]_{z_1 \cdots z_m}$ does not contain all multigraded Hilbert series.

Definition. Let σ be a monoid ordering on $\mathbb{Z}^m$.

a) An element $f \in \mathbb{Z}[[\mathbf{z}, \mathbf{z}^{-1}]]$ is called a **σ-Laurent series** if $\mathrm{Supp}\,(f)$ is well-ordered by σ.

b) The set of all σ-Laurent series forms a ring $\mathbb{Z}[[\mathbf{z}, \mathbf{z}^{-1}]]_\sigma$ which is called the **σ-Laurent series ring**.

Proposition. a) If P is weakly positively graded by W, then there exists a monoid ordering σ on $\mathbb{Z}^m$ which is a well-ordering on the set of degrees of homogeneous elements of a non-zero finitely generated graded P-module.

b) If P is positively graded by W, we can take $\sigma = \texttt{Lex}$.

Proposition. (Basic Properties of HS_M**)**

Let P be weakly positively graded by W.

a) We have $\mathrm{HS}_P(z_1, \ldots, z_m) = \frac{1}{\prod_{j=1}^n (1 - z_1^{w_{1j}} \cdots z_m^{w_{mj}})}$.

b) The shifted module $M(d)$, where $M(d)_i = M_{d+i}$ for $i \in \mathbb{Z}^m$, satisfies $\mathrm{HS}_{M(d)}(\mathbf{z}) = \mathbf{z}^{-d}\,\mathrm{HS}_M(\mathbf{z})$.

c) HS_M is additive on homogeneous short exact sequences.

d) For a graded submodule M of a graded free P-module $F = \bigoplus_{i=1}^r P(-\delta_i)$ and a module term ordering σ we have $\mathrm{HS}_M(\mathbf{z}) = \mathrm{HS}_{\mathrm{LT}_\sigma(M)}(\mathbf{z})$.

e) Under a field extension $K \subseteq L$, we have $\mathrm{HS}_{M \otimes L}(\mathbf{z}) = \mathrm{HS}_M(\mathbf{z})$.

f) The usual algorithm for computing HS_M in the standard graded case continues to work in the multigraded case.

g) The series $\mathrm{HS}_M(\mathbf{z})$ has the shape

$$\mathrm{HS}_M(\mathbf{z}) = \frac{\mathbf{z}^\alpha\ \mathrm{HN}_M(\mathbf{z})}{\prod_{j=1}^n (1 - z_1^{w_{1j}} \cdots z_m^{w_{mj}})}$$

where $\mathrm{HN}_M(\mathbf{z}) \in \mathbb{Z}[\mathbf{z}]$ is called the **multivariate Hilbert numerator** of M.

h) **(Change of Gradings)** Let $A = (a_{ij}) \in \mathrm{Mat}_{r,m}(\mathbb{Z})$ be such that $A \cdot W$ defines a weakly positive grading on P. Then we have $M_{W,d} = M_{AW,Ad}$ for all $d \in \mathbb{Z}^m$ and

$$\mathrm{HS}_{M,AW}(\mathbf{z}) = \mathrm{HS}_{M,W}(z_1^{a_{11}} \cdots z_r^{a_{r1}}, \ldots, z_1^{a_{1m}} \cdots z_r^{a_{rm}})$$

3. Hilbert Bases.

Let P be weakly positively graded by W. Let $1 \le \ell \le m$ and $A = (I_\ell\ 0) \in \mathrm{Mat}_{\ell,m-\ell}(\mathbb{Z})$. Then the matrix $U = AW$ is the *upper part* of W, i.e. W is of the form $W = \binom{U}{V}$. We say that the grading given by W is a **refinement** of the grading given by U.

Proposition. (Hilbert Series and Refinements)
Let M be a finitely generated graded P-module.

a) Let $o = (0, \ldots, 0) \in \mathbb{Z}^\ell$ and $(P_{U,o})_e = P_{(o,e)}$ for all $e \in \mathbb{Z}^{m-\ell}$. Then $P_{U,o}$ is a finitely generated graded K-algebra which is graded by V.

b) Let $(M_{U,d})_e = M_{(d,e)}$ for $d \in \mathbb{Z}^\ell$ and $e \in \mathbb{Z}^{m-\ell}$. Then $M_{U,d}$ is a finitely generated graded $P_{U,o}$-module.

c) $\mathrm{HS}_{M_{U,d},V}(\tilde{z}_1, \ldots, \tilde{z}_{m-\ell}) = \sum_{e \in \mathbb{Z}^{m-\ell}} \dim_K(M_{(d,e)})\, \tilde{z}^e$

d) If P is weakly positively graded by U, then we have $\dim_K(M_{U,d}) < \infty$ for all $d \in \mathbb{Z}^\ell$ and

$$\mathrm{HS}_{M,U}(z_1, \ldots, z_\ell) = \mathrm{HS}_{M,W}(z_1, \ldots, z_\ell, 1, \ldots, 1)$$

The most difficult part of the *proof* is a). We have

$$P_{U,o} = \langle x_1^{\alpha_1} \cdots x_n^{\alpha_n} \mid U \cdot (\alpha_1, \ldots, \alpha_n)^{\mathrm{tr}} = 0; \alpha_1, \ldots, \alpha_n \in \mathbb{N} \rangle$$

We need a description of the set of all such $(\alpha_1, \ldots, \alpha_n)$.

Definition. Let $A = (a_{ij}) \in \mathrm{Mat}_{m,n}(\mathbb{Z})$ and $\mathbf{z} = (z_1, \ldots, z_n)$.

a) The subgroup of $\mathbb{Z}^n$ of solutions of $A \cdot \mathbf{z} = 0$ is denoted by $\mathcal{L}(A)$. Morevoer, let $\mathcal{L}_+(A) = \mathcal{L}(A) \cap \mathbb{N}^n$.

b) For $j = 1, \ldots, s$, let $t_j = y_1^{a_{1j}} \cdots y_m^{a_{mj}} \in L = K[\mathbf{y}, \mathbf{y}^{-1}]$ be the **Laurent power product** associated to the j^{th} column of A.

c) Let $\varphi : P \longrightarrow L$ be defined by $\varphi(x_i) = t_i$ for $i = 1, \ldots, n$. Then $I(A) = \ker(\varphi)$ is called the **toric ideal** of A.

Proposition. (Computation of Toric Ideals)
Let $f = y_1 \cdots y_m$. For $j = 1, \ldots, n$, let $p_j = \min\{i \mid f^i\, t_j \in K[y_1, \ldots, y_m]\}$ and $t'_j = f^{p_j}\, t_j$.

a) $I(A) = [(f^{p_1} x_1 - t'_1, \ldots, f^{p_n} x_n - t'_n) : f^\infty] \cap P$

b) $I(A)$ is generated by **pure binomials**, i.e. by polynomials $\mathbf{x}^\alpha - \mathbf{x}^\beta$ such that $\gcd(\mathbf{x}^\alpha, \mathbf{x}^\beta) = 1$.

Definition. Let $A \in \mathrm{Mat}_{m,n}(\mathbb{Z})$. On $\mathcal{L}_+(A)$ we define a partial ordering $\succ$ by $(\alpha_1, \ldots, \alpha_n) \succ (\beta_1, \ldots, \beta_n) \iff$
$\iff \alpha_i \geq \beta_i$ for $i = 1, \ldots, n$.

a) The set of minimal elements of $\mathcal{L}_+(A) \setminus \{0\}$ with respect to $\succ$ is called the **Hilbert basis** of $\mathcal{L}_+(A)$.

b) The matrix $\bar{A} = \left(\begin{smallmatrix} A & 0 \\ I_n & I_n \end{smallmatrix}\right) \in \mathrm{Mat}_{m+n,2n}(\mathbb{Z})$ is called the **Lawrence lifting** of A.

Theorem. (Computation of Hilbert Bases) Let $A = (a_{ij}) \in \mathrm{Mat}_{m,n}(\mathbb{Z})$ and let G be a reduced Gröbner basis of $I(\bar{A}) \subseteq K[x_1, \ldots, x_n, w_1, \ldots, w_n]$. then the set

$$H = \{\alpha \in \mathbb{N}^n \mid \mathbf{x}^\alpha - \mathbf{w}^\alpha \in G\}$$

is the Hilbert basis of $\mathcal{L}_+(A)$.

Proposition. Let H be the Hilbert basis of $\mathcal{L}_+(A)$. Then every element of the monoid $\mathcal{L}_+(A)$ is an $\mathbb{N}$-linear combination of the elements of H.

Example. Consider $A = (3, -5, 4)$. The Lawrence lifting of A is

$$\bar{A} = \begin{pmatrix} 3 & -5 & 4 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

Using CoCoA, we find $I(\bar{A}) = (x_2x_3^2w_1 - x_1w_2w_3^2,\ x_3w_1^3w_2 - x_1^3x_2w_3,\ x_1^2x_2^2x_3 - w_1^2w_2^2w_3,\ x_3^3w_1^4 - x_1^4w_3^3,\ x_1^5x_2^3 - w_1^5w_2^3,\ x_2^4x_3^5 - w_2^4w_3^5,\ x_1x_2^3x_3^3 - w_1w_2^3w_3^3)$. Hence the Hilbert basis of $\mathcal{L}_+(A)$ is

$$\{(2, 2, 1),\ (5, 3, 0),\ (0, 4, 5),\ (1, 3, 3)\}$$

4. Application to Weakly Positive Matrices

Question: Does there exists a vector with positive coordinates which is a linear combination of the vectors $(-1, 5, 5)$ and $(-2, 13, -12)$?

In other words, is the grading defined by $W = \left(\begin{smallmatrix} -1 & 5 & 5 \\ -2 & 13 & -12 \end{smallmatrix}\right)$ weakly positive?

Answer: We need to find $(z_1, \ldots, z_5) \in \mathbb{Z}^5$ such that

$$-z_1 - 2z_2 = z_3 + 1$$

$$5z_1 + 13z_2 = z_4 + 1$$

$$-5z_1 - 12z_2 = z_5 + 1$$

and such that $z_3, z_4, z_5 \in \mathbb{N}$. Using a slight generalization of the above theory, we find the solution $(-5, 2, 0, 0, 0)$. Hence we have

$$-5\,(-1, 5, -5) + 2\,(-2, 13, -12) = (1, 1, 1)$$

i.e. the grading defined by W is weakly positive.

5. Application to Magic Squares

Question: In 1514, Albrecht Dürer painted the famous **Jupiter magic square**

$$\begin{pmatrix} 16 & 3 & 2 & 13 \\ 5 & 10 & 11 & 8 \\ 9 & 6 & 7 & 12 \\ 4 & 15 & 14 & 1 \end{pmatrix}$$

How many magic squares with **magic sum** 34 are there?

In fact, Dürer's square is **supermagic** in the sense that all 2x2 submatrices centered on the main diagonal also have magic sum 34. How many supermagic squares with magic sum 34 are there?

Answer: There are two methods for solving the first problem.

1) Using multigraded Hilbert series:

Consider a polynomial ring $P = K[y_{11}, y_{12}, \ldots, y_{44}]$ in 16 indeterminates. We equip it with the $\mathbb{Z}^{10}$-grading defined by the matrix $W =$

$$\begin{pmatrix}
1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&1&1&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&1&1&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1\\
1&0&0&0&1&0&0&0&1&0&0&0&1&0&0&0\\
0&1&0&0&0&1&0&0&0&1&0&0&0&1&0&0\\
0&0&1&0&0&0&1&0&0&0&1&0&0&0&1&0\\
0&0&0&1&0&0&0&1&0&0&0&1&0&0&0&1\\
1&0&0&0&0&1&0&0&0&0&1&0&0&0&0&1\\
0&0&0&1&0&0&1&0&0&1&0&0&1&0&0&0
\end{pmatrix}$$

A tuple $(\alpha_{11},\ldots,\alpha_{44}) \in \mathbb{N}^{16}$ defines a magic 4x4 square with magic sum s if and only if the power product $y_{11}^{\alpha_{11}} \cdots y_{44}^{\alpha_{44}}$ has multidegree $W \cdot (\alpha_{11},\ldots,\alpha_{44})^{\mathrm{tr}} = (s,s,\ldots,s)$. Therefore the number of such magic squares is the coefficient of $z_1^s \cdots z_{10}^s$ in the multigraded Hilbert series of P.

2) Using Hilbert bases:

Let a 4x4 square again be given by a tuple $(a_{11},\ldots,a_{44}) \in \mathbb{Z}^{16}$. It is magic if and only if the linear system of equations $A\,\mathbf{z} = 0$ is satisfied, where $A =$

$$\begin{pmatrix}
1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0\\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1\\
0 & 1 & 1 & 1 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0\\
1 & 0 & 1 & 1 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0\\
1 & 1 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0\\
1 & 1 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1\\
0 & 1 & 1 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1\\
1 & 1 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0
\end{pmatrix}$$

Thus we need to compute the Hilbert basis of $\mathcal{L}_+(A)$ as above. It consists of 20 magic squares $v_1, \ldots, v_{20}$, eight of which have magic sum 1 and 12 of which have magic sum 2.

In order to find the number of magic squares with magic sum 34, we compute the Hilbert function of $K[\mathrm{x}^{v_1}, \ldots, \mathrm{x}^{v_{20}}]$ in degree 34 and find that there are

$$163, 890, 864$$

of them. Similarly, we can compute that the number of supermagic squares of magic sum 34 is

$$12, 186, 468$$

6. Application to Statistics

A 2x2 **contingency table** with **margins** (b_1, b_2, c_1, c_2) is an array of natural numbers of the form

$$\begin{pmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ c_1 & c_2 & \end{pmatrix}$$

such that $b_1 = a_{11} + a_{12}$ and $b_2 = a_{21} + a_{22}$ and $c_1 = a_{11} + a_{21}$ and $c_2 = a_{12} + a_{22}$.

Question: How many contingency tables with margins $(20, 39, 27, 32)$ are there?

Answer: Again we can use either multigraded Hilbert series or Hilbert bases. For the first method, we equip the polynomial ring $P = K[x_1, x_2, x_3, x_4]$ with the $\mathbb{Z}^4$-grading defined by

$$W = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

and compute the coefficient of $z_1^{20} z_2^{39} z_3^{27} z_4^{32}$ in the multigraded Hilbert series of P. The result is 21.

For the second method, we introduce 8 indeterminates a_{ij}, b_k, c_ℓ and compute a Hilbert basis for the monoid of non-negative solutions of the above system of four equations. Using the elements of that Hilbert basis, we can actually determine the 21 solutions of our problem:

$\binom{0\ 20}{27\ 12}$, $\binom{1\ 19}{26\ 13}$, $\binom{2\ 18}{25\ 14}$, $\binom{3\ 17}{24\ 15}$, $\binom{4\ 16}{23\ 16}$, $\binom{5\ 15}{22\ 17}$, $\binom{6\ 14}{21\ 18}$, $\binom{7\ 13}{20\ 19}$, $\binom{8\ 12}{19\ 20}$, $\binom{9\ 11}{18\ 21}$, $\binom{10\ 10}{17\ 22}$, $\binom{11\ 9}{16\ 23}$, $\binom{12\ 8}{15\ 24}$, $\binom{13\ 7}{14\ 25}$, $\binom{14\ 6}{13\ 26}$, $\binom{15\ 5}{12\ 27}$, $\binom{16\ 4}{11\ 28}$, $\binom{17\ 3}{10\ 29}$, $\binom{18\ 2}{9\ 30}$, $\binom{19\ 1}{8\ 31}$, $\binom{20\ 0}{7\ 32}$