Idealization of Modules in Computer Algebra

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1. What is Idealization of Modules?

Let R be a ring and M an R-module. We turn the set $R \times M$ into a ring by componentwise addition and

$$(r, m) \cdot (r', m') = (rr', rm' + r'm)$$

This ring is called the **idealization** of M.

Remarks. a) The image of M in $R \times M$ under the injective map $\imath : M \longrightarrow R \times M$ defined by $m \mapsto (0, m)$ is an ideal.

b) The ideal $\imath(M)$ satisfies $\imath(M)^2 = 0$.

c) Given an R-submodule $N \subseteq M$, the inclusion $R \times N \subseteq R \times M$ is an injective ring homomorphism.

d) Let Γ be a monoid, let R be Γ-graded, and let M be a Γ-graded R-module. Then $R \times M$ is a Γ-graded ring via $(R \times M)_\gamma = R_\gamma \times M_\gamma$ for all $\gamma \in \Gamma$, and $\imath(M)$ is a homogeneous ideal in this ring.

2. Idealization of Graded Submodules

K field

$P = K[x_1, \ldots, x_n]$ polynomial ring

P is graded by a positive matrix $W \in \mathrm{Mat}_{m,n}(\mathbb{Z})$

$\deg_W(x_i) >_{Lex} 0$ is the i^{th} column of W

$d_{01}, \ldots, d_{0r} \in \mathbb{Z}^m$

$F_0 = \bigoplus_{i=1}^{r} P(-d_{0i})$ graded free P-module

$M \subseteq F_0$ graded submodule

$\mathcal{V} = (v_1, \ldots, v_s)$ deg-ordered tuple of non-zero homogeneous vectors which generate M

Proposition 1. (Idealization of a Free Module)
The map $\varphi : P \times F_0 \longrightarrow \overline{P}/\mathfrak{e}$ which sends $(f, (g_1, \ldots, g_r))$ to the residue class of $f + g_1 e_1 + \cdots + g_r e_r$ is an isomorphism of graded rings. Here we equip $\overline{P} = K[x_1, \ldots, x_n, e_1, \ldots, e_r]$ with the grading given by $\overline{W} = (W \mid d_{01} \ \cdots \ d_{0r})$ and $\mathfrak{e}$ is the ideal generated by

$$E = \{e_i e_j \mid 1 \leq i \leq j \leq r\} \subseteq \overline{P}$$

Proposition 2. (Idealization of a Graded Submodule) Under the composition $M \xrightarrow{\imath} P \times M \hookrightarrow P \times F_0 \xrightarrow{\varphi} \overline{P}/\mathfrak{e}$, the module M is identified with the residue class ideal of $I_M = (v_1, \ldots, v_s) + \mathfrak{e}$.

The ideal I_M is called the **(idealization) ideal** of M.

Questions. a) How do Gröbner bases behave under this process?

b) How are minimal homogeneous systems of generators of M and I_M related to each other?

c) Can the syzygy module $\mathrm{Syz}_P(\mathcal{V})$ be computed using idealization?

d) What are the advantages of using the idealization ideal I_M instead of M?

3. Gröbner Bases and Idealization

Let τ be a term ordering on $\mathbb{T}^n$, the monoid of terms in P.

Let σ be a module term ordering on $\mathbb{T}^n\langle e_1, \ldots, e_r\rangle$ which is compatible with τ, i.e. such that $t \geq_\tau t'$ implies $t\, e_i \geq_\sigma t'\, e_i$ for $i = 1, \ldots, r$.

Let $\overline{\sigma}$ be a term ordering on $\mathbb{T}(x_1, \ldots, x_n, e_1, \ldots, e_r)$ which extends both σ and τ.

Proposition 3. (Gröbner Bases and Idealization)

a) Let G be a σ-Gröbner basis of M. Then $G \cup E$ is a $\overline{\sigma}$-Gröbner basis of I_M.

b) Let G be the reduced σ-Gröbner basis of M. Then the reduced $\overline{\sigma}$-Gröbner basis of I_M is

$$G \cup \{e_i e_j \in E \mid e_i, e_j \notin \mathrm{LT}_\sigma(M)\}$$

In particular, the reduced $\overline{\sigma}$-Gröbner basis of I_M is $G \cup E$ if $M \subseteq (x_1, \ldots, x_n)\, F_0$.

4. Idealization and Minimal Homogeneous Generators

Assume that $M \subseteq (x_1, \ldots, x_n)\, F_0$.

Let $d_{0i} >_{Lex} 0$ for $i = 1, \ldots, r$. (Then $\overline{P}$ is positively graded.)

Let $\mathcal{E} = (e_1e_1, e_1e_2, \ldots, e_1e_r, e_2e_2, e_2e_3, \ldots, e_2e_r, \ldots, e_re_r)$

Proposition 4. (Idealization and Minimal Generators)

a) Let $\mathcal{V}$ be a minimal set of generators of M. Then the set $\{v_1, \ldots, v_s\} \cup E$ is a minimal set of generators of I_M.

b) If we apply Buchberger's Algorithm with Minimalization (BAM) to the tuple $(\mathcal{V} \mid \mathcal{E})$, it computes a minimal system of generators of the ideal $I_M \subseteq \overline{P}$ of the form $(\mathcal{V}_{\min} \mid \mathcal{E})$, where $\mathcal{V}_{\min}$ is a minimal set of generators of M. In particular, the elements of $\mathcal{E}$ are minimal generators of I_M.

c) If we apply the variant of BAM to the tuple $(\mathcal{V} \mid \mathcal{E})$, it computes a homogeneous $\overline{\sigma}$-Gröbner basis of I_M of the form $(\mathcal{G} \mid \mathcal{E})$ and a minimal set of generators of the form $(\mathcal{G}_{\min} \mid \mathcal{E})$, where $\mathcal{G}$ is a σ-Gröbner basis of M and $\mathcal{G}_{\min}$ is a minimal set of generators of M which is contained in $\mathcal{G}$.

The Buchberger Algorithm with Minimalization

1) Let $B = \emptyset$, $\mathcal{W} = \mathcal{V}$, $\mathcal{G} = \emptyset$, $s' = 0$, and $\mathcal{V}_{\min} = \emptyset$.

2) Let d be the smallest degree with respect to `Lex` of an element in B or in $\mathcal{W}$. Form the subset $B_d = \{(i,j) \in B \mid \deg_W(\sigma_{ij}) = d\}$ and the subtuple $\mathcal{W}_d$ of elements of degree d in $\mathcal{W}$, and delete their entries from B and $\mathcal{W}$, respectively.

3) If $B_d = \emptyset$, continue with step 6). Otherwise, choose a pair $(i,j) \in B_d$ and remove it from B_d.

4) Compute the S-vector S_{ij} and its normal remainder $S'_{ij} = \mathrm{NR}_{\sigma,\mathcal{G}}(S_{ij})$. If $S'_{ij} = 0$, continue with step 3).

5) Increase s' by one, append $g_{s'} = S'_{ij}$ to the tuple $\mathcal{G}$, and append $\{(i,s') \mid 1 \le i < s', \gamma_i = \gamma_{s'}\}$ to the set B. Continue with step 3).

6) If $\mathcal{W}_d = \emptyset$, continue with step 9). Otherwise, choose a vector $v \in \mathcal{W}_d$ and remove it from $\mathcal{W}_d$.

7) Compute $v' = \mathrm{NR}_{\sigma,\mathcal{G}}(v)$. If $v' = 0$, continue with step 6).

8) Increase s' by one, append $g_{s'} = v'$ to the tuple $\mathcal{G}$, append v to the tuple $\mathcal{V}_{\min}$, and append $\{(i,s') \mid 1 \le i < s', \gamma_i = \gamma_{s'}\}$ to the set B. Continue with step 6).

9) If $B = \emptyset$ and $\mathcal{W} = \emptyset$, return the pair $(\mathcal{G}, \mathcal{V}_{\min})$ and stop. Otherwise, continue with step 2).

This is an algorithm which returns a deg-ordered tuple $\mathcal{G} = (g_1, \ldots, g_{s'})$ whose elements are a homogeneous σ-Gröbner basis of M and a deg-ordered minimal system of generators $\mathcal{V}_{\min}$ of M which is a subtuple of $\mathcal{V}$.

The Variant of BAM

1) Let $B = \emptyset$, $\mathcal{W} = \mathcal{V}$, $\mathcal{G} = \emptyset$, $s' = 0$, and $\mathcal{G}_{\min} = \emptyset$.

2) Let d be the smallest degree with respect to `Lex` of an element in B or in $\mathcal{W}$. Form the subset $B_d = \{(i,j) \in B \mid \deg_W(\sigma_{ij}) = d\}$ and the subtuple $\mathcal{W}_d$ of elements of degree d in $\mathcal{W}$, and delete their entries from B and $\mathcal{W}$, respectively.

3) If $B_d = \emptyset$, continue with step 6). Otherwise, choose a pair $(i,j) \in B_d$ and remove it from B_d.

4) Compute the S-vector S_{ij} and its normal remainder $S'_{ij} = \mathrm{NR}_{\sigma,\mathcal{G}}(S_{ij})$. If $S'_{ij} = 0$, continue with step 3).

5) Increase s' by one, append $g_{s'} = S'_{ij}$ to the tuple $\mathcal{G}$, and append $\{(i,s') \mid 1 \le i < s', \gamma_i = \gamma_{s'}\}$ to the set B. Continue with step 3).

6) If $\mathcal{W}_d = \emptyset$, continue with step 9). Otherwise, choose a vector $v \in \mathcal{W}_d$ and remove it from $\mathcal{W}_d$.

7) Compute $v' = \mathrm{NR}_{\sigma,\mathcal{G}}(v)$. If $v' = 0$, continue with step 6).

8) Increase s' by one, append $g_{s'} = v'$ to the tuples $\mathcal{G}$ and $\mathcal{G}_{\min}$, and append $\{(i,s') \mid 1 \le i < s', \gamma_i = \gamma_{s'}\}$ to the set B. Continue with step 6).

9) If $B = \emptyset$ and $\mathcal{W} = \emptyset$, return the pair $(\mathcal{G}, \mathcal{G}_{\min})$ and stop. Otherwise, continue with step 2).

This is an algorithm which returns a deg-ordered homogeneous σ-Gröbner basis $\mathcal{G} = (g_1, \ldots, g_{s'})$ of M and a subtuple $\mathcal{G}_{\min}$ of $\mathcal{G}$ which generates M minimally.

5. Idealization of a Homogeneous Presentation

If $F_2 \xrightarrow{\psi} F_1 \xrightarrow{\varphi} M \longrightarrow 0$ is a homogeneous presentation of M and $d \in \mathbb{Z}^m$, then

$$F_2(d) \xrightarrow{\tilde{\psi}} F_1(d) \xrightarrow{\tilde{\varphi}} M(d) \longrightarrow 0$$

is a homogeneous presentation of $M(d)$. Therefore we shall from now on assume $d_{0i} >_{Lex} 0$ for $i = 1, \ldots, r$. In particular, $\overline{P}$ is then positively graded by $\overline{W}$.

$\mathcal{V} = (v_1, \ldots, v_s)$ is a deg-ordered homogeneous tuple which generates M

$d_{1i} = \deg_W(v_i) >_{Lex} 0$ for $i = 1, \ldots, s$

$F_1 = \bigoplus_{i=1}^{s} P(-d_{1i})$ graded free P-module

$\{\epsilon_1, \ldots, \epsilon_s\}$ canonical basis of F_1

$\mathrm{Syz}_P(\mathcal{V})$ is a graded submodule of F_1

$\widetilde{P} = K[x_1, \ldots, x_n, e_1, \ldots, e_r, \epsilon_1, \ldots, \epsilon_s]$ is positively graded by $\widetilde{W} = (W \mid d_{01} \cdots d_{0r} \mid d_{11} \cdots d_{1s})$.

Proposition 5. (Idealization of a Presentation)

a) The idealization of $F_0 \oplus F_1$ the ring $\widetilde{P}/\tilde{\mathfrak{e}}$, where $\tilde{\mathfrak{e}}$ is the ideal generated by $\{e_i e_j\} \cup \{e_i \epsilon_j\} \cup \{\epsilon_i \epsilon_j\}$.

b) If $(f_1, \ldots, f_s) \in \mathrm{Syz}_P(\mathcal{V})$ is a homogeneous syzygy of $\mathcal{V}$, then the corresponding element $f_1\epsilon_1 + \cdots + f_s\epsilon_s$ of $\widetilde{P}$ is contained in the ideal $(v_1 - \epsilon_1, \ldots, v_s - \epsilon_s)$.

The ideal $\tilde{I}_M = (v_1 - \epsilon_1, \ldots, v_s - \epsilon_s) + \tilde{\mathfrak{e}}$ in $\widetilde{P}$ is called the **ideal of the presentation** of M.

c) There exists a unique P-algebra homomorphism

$$\Phi : P[\epsilon_1, \ldots, \epsilon_s]/(\epsilon_i\epsilon_j)_{i,j=1,\ldots,s} \longrightarrow \overline{P}/\mathfrak{e}$$

which maps the residue class of ϵ_i to $v_i + \mathfrak{e}$ for $i = 1, \ldots, s$.

d) The image of Φ is the residue class ideal of the ideal of M.

e) The kernel of Φ is the residue class ideal of the ideal of $\mathrm{Syz}_P(\mathcal{V})$.

f) The ideal of $\mathrm{Syz}_P(\mathcal{V})$ is given by $\tilde{I}_M \cap P[\epsilon_1, \ldots, \epsilon_s]$.

6. Computing Minimal Presentations Vertically

1) Choose a term ordering σ on $\mathbb{T}^n(e_1,\dots,e_r)$. Let $B=\emptyset$, $\mathcal{W}=\mathcal{V}$, $\mathcal{G}=\emptyset$, $s'=0$, $\mathcal{G}_{\min}=\emptyset$, $\mu=0$, and $\mathcal{S}=\emptyset$.

2) Let d be the smallest degree with respect to `Lex` of an element in B or in $\mathcal{W}$. Form the subset B_d and the subtuple $\mathcal{W}_d$, and delete their entries from B and $\mathcal{W}$, respectively.

3) If $B_d=\emptyset$, continue with step 6). Otherwise, chose a pair $(i,j)\in B_d$ and remove it from B_d.

4) Compute the S-vector S_{ij} of g_i and g_j. Then compute $S'_{ij}=\mathrm{NR}_{\sigma,\mathcal{G}}(S_{ij})$. If $S'_{ij}=0$ continue with step 3). If $S'_{ij}\neq 0$ and it does not involve the indeterminates $e_1,\dots,e_r$, then append it to $\mathcal{S}$ and continue with step 3).

5) Increase s' by one, append $g_{s'}=S'_{ij}$ to the tuple $\mathcal{G}$, and append the set $\{(i,s')\mid 1\le i<s',\ \gamma_i=\gamma_{s'}\}$ to B. Then continue with step 3).

6) If $\mathcal{W}_d=\emptyset$, continue with step 9). Otherwise, choose a vector $v\in\mathcal{W}_d$ and remove it from $\mathcal{W}_d$.

7) Compute $v'=\mathrm{NR}_{\sigma,\mathcal{G}}(v)$ and $\bar{v}=v'|_{\epsilon_j\mapsto 0}$. If $\bar{v}=0$, continue with step 6).

8) Increase s' and μ by one. Adjoin a new indeterminate ϵ_μ to $\overline{P}$ and extend the grading to this new ring by defining $\deg_{\overline{W}}(\epsilon_\mu)=\deg_{\overline{W}}(\bar{v})$. Extend the term ordering σ to the new ring in such a way that the extension is an elimination ordering for $\{e_1,\dots,e_r\}$. Append $g_{s'}=\bar{v}-\epsilon_\mu$ to $\mathcal{G}$ and $\bar{v}$ to $\mathcal{G}_{\min}$. Append the set $\{(i,s')\mid 1\le i<s',\gamma_i=\gamma_{s'}\}$ to B. Continue with step 6).

9) If $B \neq \emptyset$ or $\mathcal{W} \neq \emptyset$, continue with step 2).

10) Apply the Buchberger Algorithm with Minimalization to the module generated by $\mathcal{S}$ and obtain a subtuple $\mathcal{S}_{\min}$ of $\mathcal{S}$ which minimally generates that module. Return the pair $(\mathcal{G}_{\min}, \mathcal{S}_{\min})$ and stop.

This is an algorithm which computes a pair $(\mathcal{G}_{\min}, \mathcal{S}_{\min})$ such that $\mathcal{G}_{\min}$ is a deg-ordered tuple of homogeneous vectors in F_0 which generate M minimally, and such that $\mathcal{S}_{\min}$ is a deg-ordered tuple of homogeneous vectors in $\bigoplus_{i=1}^{\mu} P(-d_{1i})$ which generate $\mathrm{Syz}_P(\mathcal{G}_{\min})$ minimally.

Here μ is the number of elements in $\mathcal{G}_{\min}$ and d_{1i} is the degree of the i^{th} element in $\mathcal{G}_{\min}$ for $i = 1, \ldots, \mu$.

7. Computing Minimal Presentations Horizontally

1) Choose a term ordering σ on $\mathbb{T}^n(e_1,\dots,e_r)$. Let $B=\emptyset$, $B'=\emptyset$, $\mathcal{W}=\mathcal{V}$, $\mathcal{G}=\emptyset$, $s'=0$, $\mathcal{G}_{\min}=\emptyset$, $\mu=0$, $\mathcal{S}=\emptyset$, $s''=0$, and $\mathcal{S}_{\min}=\emptyset$.

2) Let d be the smallest degree with respect to `Lex` of an element in $B\cup B'$ or in $\mathcal{W}$. Form the subset B_d of B, the subset B'_d of B', the subtuple $\mathcal{W}_d$ of $\mathcal{W}$, and delete their entries from B, B', and $\mathcal{W}$, respectively.

3) If $B'_d=\emptyset$, continue with step 6). Otherwise, choose a pair $(i,j)\in B'_d$ and remove it from B'_d.

4) Compute the S-vector of h_i and h_j and call it S_{ij}. Then compute the normal remainder $S'_{ij}=\mathrm{NR}_{\sigma,\mathcal{S}}(S_{ij})$. If $S'_{ij}=0$, continue with step 3).

5) Increase s'' by one, append $h_{s''}=S'_{ij}$ to the tuple $\mathcal{S}$, append the set $\{(i,s'')\mid 1\le i<s'',\ \eta_i=\eta_{s''}\}$ to B', and continue with step 3).

6) If $B_d=\emptyset$, continue with step 10). Otherwise, choose a pair $(i,j)\in B_d$ and remove it from B_d.

7) Compute the S-vector of g_i and g_j and call it S_{ij}. Then compute the normal remainder $S'_{ij}=\mathrm{NR}_{\sigma,\mathcal{G}}(S_{ij})$. If S'_{ij} involves one of the indeterminates $e_1,\dots,e_r$, then increase s' by one, append $g_{s'}=S'_{ij}$ to $\mathcal{G}$, append $\{(i,s')\mid 1\le i< s',\gamma_i=\gamma_{s'}\}$ to B, and continue with step 6).

8) Compute the normal remainder $S''_{ij}=\mathrm{NR}_{\sigma,\mathcal{S}}(S'_{ij})$. If $S''_{ij}=0$, continue with step 6).

9) Increase s'' by one. Append $h_{s''} = S''_{ij}$ to the tuples $\mathcal{S}$ and $\mathcal{S}_{\min}$. Append $\{(i, s'') \mid 1 \le i < s'',\ \eta_i = \eta_{s''}\}$ to B'. Continue with step 6).

10) If $\mathcal{W}_d = \emptyset$, continue with step 13). Otherwise, choose a vector $v \in \mathcal{W}_d$ and remove it from $\mathcal{W}_d$.

11) Compute $v' = \mathrm{NR}_{\sigma,\mathcal{G}}(v)$ and $\bar{v} = v'|_{\epsilon_j \mapsto 0}$. If $\bar{v} = 0$, continue with step 10).

12) Increase s' and μ by one. Adjoin a new indeterminate ϵ_μ to $\overline{P}$ and extend the grading to this new ring by defining $\deg_{\overline{W}}(\epsilon_\mu) = \deg_{\overline{W}}(v')$. Extend the term ordering σ to the new ring in such a way that the extension is an elimination ordering for $\{e_1, \ldots, e_r\}$. Append $g_{s'} = \bar{v} - \epsilon_\mu$ to the tuple $\mathcal{G}$ and $\bar{v}$ to $\mathcal{G}_{\min}$. Append $\{(i, s') \mid 1 \le i < s',\ \gamma_i = \gamma_{s'}\}$ to B. Continue with step 10).

13) If $B = B' = \emptyset$ and $\mathcal{W} = \emptyset$, then return the pair $(\mathcal{G}_{\min}, \mathcal{S}_{\min})$ and stop. Otherwise, continue with step 2).

This is an algorithm which computes a pair $(\mathcal{G}_{\min}, \mathcal{S}_{\min})$ such that $\mathcal{G}_{\min}$ is a deg-ordered tuple of homogeneous vectors in F_0 which generate M minimally, and such that $\mathcal{S}_{\min}$ is a deg-ordered tuple of homogeneous vectors in $\bigoplus_{i=1}^{\mu} P(-d_{1i})$ which generate $\mathrm{Syz}_P(\mathcal{G}_{\min})$ minimally.

Here μ is the number of elements in $\mathcal{G}_{\min}$ and d_{1i} is the degree of the i^{th} element in $\mathcal{G}_{\min}$ for $i = 1, \ldots, \mu$.

8. Idealization of Minimal Graded Free Resolutions

Let $\mathcal{V} = (v_1, \ldots, v_s)$ be a deg-ordered tuple of homogeneous vectors generating a P-submodule M of $F_0 = \bigoplus_{i=1}^{r_0} P(-d_{0i})$. The graded free resolution of M has the shape

$$0 \longrightarrow F_\ell \xrightarrow{\varphi_\ell} F_{\ell-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{\varphi_1} F_0 \xrightarrow{\varphi_0} M \longrightarrow 0$$

where $F_0, \ldots, F_\ell$ are finitely generated graded free P-modules and for every $i \in \{1, \ldots, \ell\}$ the homogeneous homomorphism φ_i maps the canonical basis of F_i to a minimal homogeneous system of generators of $\operatorname{Ker}(\varphi_{i-1})$.

Recall that $\ell \le n$.

We may assume that $d_{0i} >_{Lex} 0$ for $i = 1, \ldots, r_0$.

We write $F_i = \bigoplus_{j=1}^{r_i} P(-d_{ij})$.

The canonical basis $\{\epsilon_1^{(i)}, \ldots, \epsilon_{r_i}^{(i)}\}$ of F_i is always kept deg-ordered.

Remarks. a) The idealization of $F_0 \oplus F_1 \oplus \cdots \oplus F_n$ is a residue class ring of the polynomial ring

$$\widetilde{P} = K[\{\epsilon_j^{(i)} \mid i \in \{0, \ldots, n\}, j \in \{1, \ldots, r_i\}\}]$$

b) The elements of F_i can be identified with their images in $\widetilde{P}$.

9. Computing Minimal Resolutions Vertically

1) Let $i = 0$. Equip $\overline{P} = P[\epsilon_1^{(i)}, \ldots, \epsilon_{r_i}^{(i)}]$ with the grading defined by $\overline{W} = (W \mid d_{i1} \ \cdots \ d_{ir_i})$. Choose a term ordering σ on $\mathbb{T}^n(\epsilon_1^{(i)}, \ldots, \epsilon_{r_i}^{(i)})$. Let $B = \emptyset$, $\mathcal{W} = \mathcal{V}$, $\mathcal{G} = \emptyset$, $s' = 0$, $\mathcal{G}_{\min}^{(i)} = \emptyset$, $r_{i+1} = 0$, and $\mathcal{S} = \emptyset$.

2) Let d be the smallest degree with respect to `Lex` of an element in B or in $\mathcal{W}$. Form the subset B_d and the subtuple $\mathcal{W}_d$, and delete their entries from B and $\mathcal{W}$, respectively.

3) If $B_d = \emptyset$, continue with step 6). Otherwise, chose a pair $(j, k) \in B_d$ and remove it from B_d.

4) Form the S-vector S_{jk} of g_j and g_k. Then compute $S'_{jk} = \mathrm{NR}_{\sigma,\mathcal{G}}(S_{jk})$. If $S'_{jk} = 0$ continue with step 3). If $S'_{jk} \neq 0$ and it does not involve the indeterminates $\epsilon_1^{(i)}, \ldots, \epsilon_{r_i}^{(i)}$, then append it to $\mathcal{S}$ and continue with step 3).

5) Increase s' by one, append $g_{s'} = S'_{jk}$ to the tuple $\mathcal{G}$, and append the set $\{(j, s') \mid 1 \le j < s',\ \gamma_j = \gamma_{s'}\}$ to B. Then continue with step 3).

6) If $\mathcal{W}_d = \emptyset$, continue with step 9). Otherwise, choose a vector $v \in \mathcal{W}_d$ and remove it from $\mathcal{W}_d$.

7) Compute $v' = \mathrm{NR}_{\sigma,\mathcal{G}}(v)$ and $\bar{v} = v'|_{\epsilon_j^{(i+1)} \mapsto 0}$. If $\bar{v} = 0$, continue with step 6).

8) Increase s' and r_{i+1} by one. Adjoin a new indeterminate $\epsilon_{r_{i+1}}^{(i+1)}$ to $\overline{P}$ and extend the grading to this new ring by defining $\deg_{\overline{W}}(\epsilon_{r_{i+1}}^{(i+1)}) = \deg_{\overline{W}}(\bar{v})$. Extend the term ordering σ to the new ring in such a way that the extension is an elimination ordering for $\{\epsilon_1^{(i)}, \ldots, \epsilon_{r_i}^{(i)}\}$. Append the element

$g_{s'} = \bar{v} - \epsilon_{r_{i+1}}^{(i+1)}$ to $\mathcal{G}$ and the element $\bar{v}$ to $\mathcal{G}_{\min}^{(i)}$. Append the set $\{(j, s') \mid 1 \leq j < s',\, \gamma_j = \gamma_{s'}\}$ to B. Continue with step 6).

9) If $B \neq \emptyset$ or $\mathcal{W} \neq \emptyset$, continue with step 2).

10) If $\mathcal{S} \neq 0$, then increase i by one and equip $\overline{P} = P[\epsilon_1^{(i)}, \ldots, \epsilon_{r_i}^{(i)}]$ with the grading defined by the matrix $\overline{W} = (W \mid d_{i1} \; \cdots \; d_{ir_i})$. Restrict σ to $\mathbb{T}^n(\epsilon_1^{(i)}, \ldots, \epsilon_{r_i}^{(i)})$. Let $B = \emptyset$, $\mathcal{W} = \mathcal{S}$, $\mathcal{G} = \emptyset$, $s' = 0$, $\mathcal{G}_{\min}^{(i)} = \emptyset$, $r_{i+1} = 0$, and $\mathcal{S} = \emptyset$. Then continue with step 2).

11) Let $\ell = i + 1$. Return the list $(\mathcal{G}_{\min}^{(0)}, \ldots, \mathcal{G}_{\min}^{(\ell-1)})$ and stop.

This is an algorithm which computes a list of deg-ordered homogeneous matrices $(\mathcal{G}_{\min}^{(0)}, \ldots, \mathcal{G}_{\min}^{(\ell-1)})$ such that the P-linear maps $\varphi_j : F_j \longrightarrow F_{j-1}$ given by $\mathcal{G}_{\min}^{(j-1)}$ for $j = 1, \ldots, \ell$ yield a minimal graded free resolution

$$0 \longrightarrow F_\ell \xrightarrow{\varphi_\ell} F_{\ell-1} \xrightarrow{\varphi_{\ell-1}} \cdots \xrightarrow{\varphi_3} F_2 \xrightarrow{\varphi_2} F_1 \xrightarrow{\varphi_1} M \longrightarrow 0$$

9. Computing Minimal Resolutions Horizontally

1) Let σ be a term ordering on $\mathbb{T}^n(\epsilon_1^{(0)}, \ldots, \epsilon_{r_0}^{(0)})$, let $\overline{P} = P[\epsilon_1^{(0)}, \ldots, \epsilon_{r_0}^{(0)}]$ be graded by $\overline{W} = (W \mid d_{01} \ \cdots \ d_{0r_0})$, let $r_1 = \cdots = r_n = 0$, let $B = \{v_1, \ldots, v_s\}$, let $\mathcal{G} = \emptyset$, and let $\mathcal{G}_{\min} = \emptyset$.

2) Let d be the smallest degree with respect to `Lex` of an element of B. Form the subset B_d of B and remove it from B.

3) If $B_d = \emptyset$, continue with step 7). Otherwise, let i be the largest upper index of an indeterminate $\epsilon_k^{(j)}$ occuring in a polynomial of B_d. Let $f \in B_d$ be a polynomial which involves that indeterminate. Remove f from B_d.

4) Compute $f' = \mathrm{HR}_{\sigma,\mathcal{G}}(f)$. If one of the indeterminates $\epsilon_1^{(i-1)}, \ldots, \epsilon_{r_{i-1}}^{(i-1)}$ occurs in f', append f' to $\mathcal{G}$, append to B all S-polynomials of f' and a polynomial g in $\mathcal{G}$ such that $\mathrm{LT}_\sigma(f')$ and $\mathrm{LT}_\sigma(g)$ involve the same $\epsilon_j^{(i-1)}$, and continue with step 3).

5) If none of the indeterminates $\{\epsilon_1^{(i)}, \ldots, \epsilon_{r_i}^{(i)}\}$ occurs in f', continue with step 3).

6) Increase r_{i+1} by one. Adjoin an indeterminate $\epsilon_{r_{i+1}}^{(i+1)}$ to $\overline{P}$ and extend the grading to this new ring by defining $\deg_{\overline{W}}(\epsilon_{r_{i+1}}^{(i+1)}) = d$. Extend the term ordering σ to the new ring in such a way that the extension is an elimination ordering for $\{\epsilon_1^{(0)}, \ldots, \epsilon_{r_i}^{(i)}\}$. Compute the polynomial

$$\overline{f} = f'(x_1, \ldots, x_n, \epsilon_1^{(i)}, \ldots, \epsilon_{r_i}^{(i)}, 0, \ldots, 0)$$

Append $g = \overline{f} - \epsilon_{r_{i+1}}^{(i+1)}$ to $\mathcal{G}$ and $\overline{f}$ to $\mathcal{G}_{\min}$. For all $h \in B$ such that $\mathrm{LT}_\sigma(g)$ and $\mathrm{LT}_\sigma(h)$ involve the same indeterminate $\epsilon_j^{(i)}$, compute the S-polynomial of g and h and append it to B. Then continue with step 3).

7) If $B = \emptyset$, return the tuple $\mathcal{G}_{\min}$ and stop. Otherwise, continue with step 2).

This is an algorithm which computes a deg-ordered tuple $\mathcal{G}_{\min}$ of homogeneous polynomials in

$$\overline{P} = P[\epsilon_1^{(0)}, \ldots, \epsilon_{r_0}^{(0)}, \ldots, \epsilon_1^{(\ell)}, \ldots, \epsilon_{r_\ell}^{(\ell)}]$$

such that the homogeneous maps of graded free P-modules $\varphi_i : F_i \longrightarrow F_{i-1}$ defined by the elements of $\mathcal{G} \cap P[\epsilon_1^{(i)}, \ldots, \epsilon_{r_i}^{(i)}]$ yield a minimal graded free resolution

$$0 \longrightarrow F_\ell \xrightarrow{\varphi_\ell} F_{\ell-1} \xrightarrow{\varphi_{\ell-1}} \cdots \xrightarrow{\varphi_3} F_2 \xrightarrow{\varphi_2} F_1 \xrightarrow{\varphi_1} M \longrightarrow 0$$

Summary and Conclusions

• For submodules of graded free modules the presentation of the idealization was described explicitly.

• We have determined the relation between the module and its idealization with respect to their Gröbner bases and with respect to minimal homogeneous sets of generators.

• We can idealize a homogeneous presentation and even a graded free resolution of the module.

• The computation of a minimal homogeneous presentation or a graded free resolution is then nothing but the computation of one Gröbner basis for the ideal representing the idealization.

• Classical strategies for computing resolutions correspond to different selection strategies for this Gröbner basis computation.

• The horizontal strategy corresponds to a particularly brief algorithm.

• All standard optimizations (avoiding unnecessary critical pairs, Hilbert driven, ...) can be applied.

• It is easy to analyze which operations have to be implemented efficiently to speed up the computation.

If you can't realize your ideal,
idealize the real.
(Marriage Counsel)