Best Approximation to an Element of an Inner Product Space from the Range of a Linear Operator over a Polyhedron

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Abstract

The range of the linear operator over a polyhedron considered in this paper is a convex subset $\mathbb{H}$ of a finite dimensional subspace $\mathbb{S}$ of the ambient inner product space $\mathbb{X}$. According to the reduction principle the best approximation to a point of $\mathbb{X}$ from $\mathbb{H}$ equals the best approximation to the best approximation to that point of $\mathbb{X}$ from $\mathbb{S}$ from $\mathbb{H}$. Equivalent representations are presented for $\mathbb{H}$ in terms of 'opposite' translated convex cones. Explicit calculations are given for the best approximations from finitely generated, translated convex cones. An interesting special case is treated when such a best approximation is to a point in a - in this paper introduced - *translated polar cone*. Several results are presented to establish the position of the projection of a point of $\mathbb{X}$ onto $\mathbb{S}$ with respect to the set $\mathbb{H}$. Using the Boyle-Dykstra theorem, it is proven that these best approximations are precisely the ones from the set $\mathbb{H}$ itself. The result is finally applied to real-life data from oil industry in that a solution is presented for a very important problem in oil - and gas production operations called the reconciliation problem, where the contribution of individual wells to a measured total production has to be assessed.

1 Introduction

We start by formulating the problem addressed in this paper.
A computable solution is presented for the following problem:

Given a real inner product space $\mathbb{X}$ *with inner product* $\langle \cdot , \cdot \rangle : \mathbb{X} \times \mathbb{X} \to \Re$, *given a finite set of linearly independent elements of* $\mathbb{X}$

$$\mathbb{Y} = \{ y_1, \ldots, y_n \} \tag{1}$$

and given the collections of real numbers $\alpha_1, \ldots, \alpha_n$ *and* $\beta_1, \ldots, \beta_n$ *with* $\alpha_i < \beta_i$ *and given the set*

$$\Gamma = \{ \chi \in \Re^n \mid (\chi)_i \in [\alpha_i, \beta_i] \ , \ i = 1, \ldots, n \} \tag{2}$$

with $[\alpha_i, \beta_i]$ *a closed interval in* $\Re$. *Consider the linear operator*

$$\mathbf{Q}_{\mathbb{Y}} : \Gamma \to \mathbb{X}$$

defined by

$$\gamma \mapsto \sum_{i=1}^{n} \gamma_i y_i$$

where $\gamma_i = (\gamma)_i$*, i.e. the* i^{th} *coordinate of* γ *in* $\Re^n$*. Then the range of* $\mathbf{Q}_{\mathbb{Y}}$ *is:*

$$\mathbf{Q}_{\mathbb{Y}}(\Gamma) := \mathbb{H} = \{ \, y \in \mathbb{X} \mid y = \sum_{i=1}^{n} \gamma_i y_i \; , \; y_i \in \mathbb{Y} \; , \; \gamma \in \Gamma \, \} \tag{3}$$

Find the best approximation - see definition 1.1 below - to an element $x \in \mathbb{X}$ *from* $\mathbb{H}$*.*

Following F. Deutsch [2], the *best approximation* is defined in the following way:

Definition 1.1 *Let* $\mathbb{A}$ *be a nonempty subset of the inner product space* $\mathbb{X}$*, and let* $x \in \mathbb{X}$*. An element* $a_0 \in \mathbb{A}$ *is called a best approximation to* x *from* $\mathbb{A}$ *if* $\|x - a_0\| = \inf_{a \in \mathbb{A}} \|x - a\|$ *, where* $\| \cdot \|$ *is the norm induced by the inner product on* $\mathbb{X}$*.*
The set of all best approximations to $x \in \mathbb{X}$ *from* $\mathbb{A}$ *is denoted by* $\mathbf{P}_{\mathbb{A}}(x)$*. The mapping* $\mathbf{P}_{\mathbb{A}}$ *from* $\mathbb{X}$ *into the subsets of* $\mathbb{A}$ *is called the metric projection onto* $\mathbb{A}$*. If each* $x \in \mathbb{X}$ *has exactly one best approximation in* $\mathbb{A}$*, i.e.* $\mathbf{P}_{\mathbb{A}}(x)$ *is a singleton, then* $\mathbb{A}$ *is called a Chebyshev set.*

The following comments on the above problem statement are in order:

- The parameter set Γ in equation (2) is a *polyhedron* in $\Re^n$ - see R. Webster [7].
- Clearly the set $\mathbb{H}$ in equation (3) is a *closed, convex* subset in the inner product space $\mathbb{X}$. The polyhedral structure of Γ in $\Re^n$ is passed on to $\mathbb{H}$ in $\mathbb{X}$ through the mapping $\mathbf{Q}_{\mathbb{Y}}$; this mapping is, specifically in the theory of frames, called the *pre-frame operator* - see O. Christensen [1].
- The linear independence assumption for the set $\mathbb{Y}$ in equation (1) is not a restriction in the sense that in case the set $\mathbb{Y}$ is linearly dependent, the problem can be reformulated in terms of the maximal - in terms of cardinality ($| \cdot |$) - independent subset $\mathbb{Y}^*$ of $\mathbb{Y}$, by adapting the parameters α_i and β_i ($i = 1, \ldots, |\mathbb{Y}^*|$) from the original problem formulation to cater for the linear dependence, and setting the rest of these parameters to zero, where without loss of generality we have assumed that the first $|\mathbb{Y}^*|$ elements of $\mathbb{Y}$ form a maximal, linearly independent set.

- We do realize that a 'toolbox solution' for the above problem exists, i.e. our problem can be formulated as a *constrained optimization* problem. The motivation for following our analytical approach reads as follows:

  1. Explicit computations of approximations from *closed convex sets*, including *closed convex cones* are sparse in the literature, as opposed to those from - translates of - subspaces - see F. Deutsch [2] and J-B. Hiriart-Urruty and C. Lemaréchal [5]. Our contribution is that we give explicit computations of the metric projection on *finitely generated convex cones*, and these results are used subsequently to compute the best approximation from the convex set $\mathbb{H}$. These results are therefore of independent interest.
  2. In addition to the previously mentioned contribution, new results are presented concerning the location determination of points with respect to convex sets, which is of crucial importance for an efficient calculation of the metric projection of those points onto this convex set. Also these results are felt to be of independent interest.
  3. Our result requires some *direct computations*, notably to accomplish the location determination of a point with respect to the set $\mathbb{H}$, but it does *not rely on searching*. We show - in section 4 - that when optimal use is made of the gemetric information contained in our problem setting, these direct computations are of *linear order in* n only. Our result may therefore compete from the computational point of view with the toolbox methods, which rely completely on searching.
  4. In applications the elements comprising our problem statement have in general a particular physical significance. The analytical approach is in this vein absolutely superior to the toolbox method, since it offers the possibility both to assess and to apply in-between results in relation with the application under consideration. We substantiate this claim our the final section.

The rest of this paper is organized as follows: in the next section we explore the geometry of the best approximation problem investigated here, followed by the explicit computation of metric projections onto elements of this geometric setting, which are moreover of interest in their own right. These results are the combined to give the best approximation from the convex set $\mathbb{H}$. In the final section we demonstrate the applicability of our result to real-life problems by presenting a solution for an oil well production allocation problem from oil industry.

The element $x \in \mathbb{X}$, the space $\mathbb{X}$, and the sets $\mathbb{Y}$, $\mathbb{H}$, Γ and the real numbers $\alpha_1, \ldots, \alpha_n$ and $\beta_1, \ldots, \beta_n$ are fixed throughout this paper, and so we avoid unnecessary repetitions of definitions of these objects in the different statements that follow. Moreover two subsets of natural numbers will be used repeatedly in this paper, and for this reason we give them a name:

$$\mathbb{I} := \{1, \ldots, n\} \ , \ \mathbb{J} := \{1, \ldots, 2^n\}$$

2 The Geometric Structure of the Projection Problem

A characterization of best approximations from convex sets in an inner product space is stated and proved in F. Deutsch [2]:

Theorem 2.1 *Let $\mathbb{A}$ be a convex set in $\mathbb{X}$, and let $a_0 \in \mathbb{A}$. Then*

$$a_0 = \mathbf{P}_{\mathbb{A}}(x) \Leftrightarrow \langle x - a_0 \, , \, a - a_0 \rangle \leq 0 \quad \forall a \in \mathbb{A} \qquad \square$$

So to compute the best approximation from the set $\mathbb{H}$, this theorem implies in any case that we must have a possibility to address all elements of $\mathbb{H}$. For this to be possible at all we need to create some reference frame on $\mathbb{H}$, or rather to embed $\mathbb{H}$ in a convenient frame. The most straightforward way to achieve this would seem to consider the *linear span* of $\mathbb{H}$:

Definition 2.1

$$\mathbb{S} := span(\mathbb{H}) = span(\mathbb{Y}) = \{ y \in \mathbb{X} \mid y = \sum_{i=1}^{n} \tau_i y_i \, , \, y_i \in \mathbb{Y} \, , \, \tau_i \in \Re \, , \, n \in \mathfrak{N} \} \quad (4)$$

That $\mathbb{S}$ is really a convenient environment for $\mathbb{H}$ specifically in relation with finding the metric projection onto it, is substantiated in the following result taken from F. Deutsch [2]:

Theorem 2.2

(1) Every closed, convex subset of $\mathbb{S}$ is Chebyshev.

(2) $\mathbb{S}$ itself is Chebyshev. $\square$

So in particular $\mathbb{H}$ as a closed, convex subset of $\mathbb{S}$, is Chebyshev, i.e. $\mathbf{P}_{\mathbb{H}}(x)$ is a singleton. This invites to a best approximation route to $x \in \mathbb{X}$ from $\mathbb{H}$ via $\mathbb{S}$. But the question is of course whether this detour will produce $\mathbf{P}_{\mathbb{H}}(x)$ at all. The following result shows that the answer to this question is in the affirmative:

Theorem 2.3 ***(The Reduction Principle)*** *Let $\mathbb{A}$ be a convex subset of $\mathbb{S}$. Then*

$$\mathbf{P}_{\mathbb{A}}(x) = \mathbf{P}_{\mathbb{A}}(\mathbf{P}_{\mathbb{S}}(x)) \qquad \square$$

For a proof of this result, we refer again to F. Deutsch [2]. So to compute the best approximation from $\mathbb{H}$, we compute the metric projection onto $\mathbb{S}$ first - that should not be a problem, since $\mathbb{S}$ is a subspace, and moreover, according to theorem 2.2 Chebyshev - and subsequently project $\mathbf{P}_{\mathbb{S}}(x)$ onto $\mathbb{H}$.

Now that we have found apparently a convenient reference frame in which $\mathbb{H}$ is embedded, there is one problem still open, namely addressing all elements of $\mathbb{H}$ to check the inequality in the best approximation characterization theorem 2.1, whereas a new one has entered the stage, namely are there, from a geometric point of view different locations of $\mathbf{P}_{\mathbb{S}}(x)$ with respect to $\mathbb{H}$, since these differences may have their bearings on the computation of $\mathbf{P}_{\mathbb{H}}(x)$ according to the reduction principle, theorem 2.3.

To start with the first question, clearly what we would need is the involvement of *finiteness*, or *finite generation* in terms of the elements y_i of the set $\mathbb{Y}$ for the description of $\mathbb{H}$. We will show that it is possible to give a geometric description of $\mathbb{H}$ in terms of, in the sense indicated above, finitely generated components. However, we need to introduce first a number of objects before stating this result. The following definition, taken from F. Deutsch [2] is important in connection with the shape of $\mathbb{H}$:

Definition 2.2 *$h \in \mathbb{H}$ is an extreme point of $\mathbb{H}$, if $f, g \in \mathbb{H}$, $0 < \lambda < 1$, and $h = \lambda f + (1-\lambda)g$ implies $f = g = h$.*

With reference to this definition, the *set* $\mathbb{E}$ *of extreme points of* $\mathbb{H}$ is introduced:

$$\mathbb{E} = \{r^1, \ldots, r^{2^n} \mid r^j = \sum_{i=1}^{n} \psi_i^j y_i \, ; \, y_i \in \mathbb{Y} \, , \, \psi_i^j = (\alpha_i \, \vee \, \beta_i) \, , \, j \in \mathbb{J} \, , \\ \psi_i^j \neq \psi_i^k \, (j \neq k) \, \} \tag{5}$$

Of special importance are the *pairs of opposite extreme points* of $\mathbb{H}$, as well as the concept of an *adjacent extreme point* of an extreme point:

Definition 2.3 *$\{r^j, r^k\}$ $(r^j \, , \, r^k \in \mathbb{E} \, , \, j \neq k \,)$ is a pair of opposite extreme points of $\mathbb{H}$ if $\psi_i^j \neq \psi_i^k \; \forall \, i$. Without loss of generality it may be assumed that the set $\mathbb{E}$ is ordered in such a way that $\{\{r^{2j-1}, r^{2j}\} \mid j = 1, \ldots, 2^{n-1}\}$ are the opposite pairs of $\mathbb{E}$, for instance $\psi_i^1 = \alpha_i \, , \, \psi_i^2 = \beta_i \, , \ldots$, and that is order is fixed.*

$r^k \in \mathbb{E}$ is an adjacent extreme point of r^j if $\begin{cases} \psi_i^k \neq \psi_i^j \textit{ for exactly one } i \in \mathbb{I} \\ \psi_i^k = \psi_i^j \textit{ for all other } i \in \mathbb{I} \end{cases}$

The following important geometric object can be associated with an extreme point $r^j \in \mathbb{E}$:

Definition 2.4 *The translated conical hull associated with the extreme point* $r^j \in \mathbb{E}$ *is, accepting a slight notational abuse - see below - given by*

$$\mathbb{K}^j := r^j + con(\mathbb{Z}^j) \tag{6}$$

where

$$\mathbb{Z}^j = \{z_1^j, \ldots, z_n^j \mid z_i^j = \begin{cases} y_i & \mathit{if}\, \psi_i^j = \alpha_i \\ -y_i & \mathit{if}\, \psi_i^j = \beta_i \end{cases} \} \tag{7}$$

$$r^j = \sum_{i=1}^{n} \sigma_i^j \psi_i^j z_i^j = \sum_{i=1}^{n} \sigma_i^k \psi_i^j z_i^k$$

$$\sigma_i^k = sgn(\alpha_i - \psi^k)\,,\ (k \in \mathbb{J}\,,\ i \in \mathbb{I}) \tag{8}$$

and for $\rho \in \Re$

$$sgn(\rho) = \begin{cases} 1 & \mathit{if}\, \rho \geq 0 \\ -1 & \mathit{if}\, \rho < 0 \end{cases}$$

and

$$con(\mathbb{Z}^j) = \{\, y \in \mathbb{X} \mid\ y = \sum_{i=1}^{n} \tau_i z_i^j\,,\ z_i^j \in \mathbb{Z}^j\,,\ \tau_i \in \Re^+ \cup \{0\}\,\}$$

is called the conical hull of $\mathbb{Z}^j$*.*

The notational abuse noted in definition 2.4 refers to the fact that equation (6) should be interpreted as a direct set sum

$$\mathbb{K}^j = \{r^j\} + con(\mathbb{Z}^j)$$

We are consistent throughout this paper in this notational abuse with respect to all by a singleton translated sets.
Following F. Deutsch [2], $con(\mathbb{Z}^j)$ is called a *finitely generated* convex cone because it is the conical hull of a finite number of vectors. The importance of the translated conical hull in the present context follows from the following proposition:

Proposition 2.1 $y + con(\mathbb{Z}^j)$ *is Chebyshev for any* $y \in \mathbb{S}$*.*

Proof: If $con(\mathbb{Z}^j)$ is Chebyshev, then so is its translate - see [2]. So in view of the first part of theorem 2.2 it only needs to be shown that $con(\mathbb{Z}^j)$ is closed in $\mathbb{S}$; this is done in [2]. □

The next proposition gives the decompositions of $\mathbb{H}$ in terms of finitely generated, translated convex cones:

Proposition 2.2 *Let* $\{r^{2j-1}, r^{2j}\}$ $(r^{2j-1}, r^{2j} \in \mathbb{E})$ *be any pair of opposite extreme points of* $\mathbb{H}$*. Then*

$$\begin{aligned} r^{2j} &\in r^{2j-1} + con(\mathbb{Z}^{2j-1}) \\ r^{2j-1} &\in r^{2j} + con(\mathbb{Z}^{2j}) \\ \mathbb{H} &= (r^{2j-1} + con(\mathbb{Z}^{2j-1})) \cap (r^{2j} + con(\mathbb{Z}^{2j})) \end{aligned}$$

Proof: It suffices to proof only the last statement.

$$\begin{aligned}
\mathbb{H} &= \{y \in \mathbb{X} \ \mid \ y = \sum_{i=1}^{n} \gamma_i y_i \, , \, \alpha_i \leq \gamma_i \leq \beta_i\} \\
&= \{y \in \mathbb{X} \ \mid \ y = \sum_{i=1}^{n} (\sigma_i^{2j-1}\gamma_i) z_i^{2j-1} \, , \, \sigma_i^{2j-1}\gamma_i \geq \sigma_i^{2j-1}\psi_i^{2j-1}\} \cap \\
&\quad \{y \in \mathbb{X} \ \mid \ y = \sum_{i=1}^{n} (\sigma_i^{2j}\gamma_i) z_i^{2j} \, , \, \sigma_i^{2j}\gamma_i \geq \sigma_i^{2j}\psi_i^{2j}\}
\end{aligned}$$

$$\begin{aligned}
&\{y \in \mathbb{X} \ \mid \ y = \sum_{i=1}^{n} (\sigma_i^{2j-1}\gamma_i) z_i^{2j-1} \, , \, \sigma_i^{2j-1}\gamma_i \geq \sigma_i^{2j-1}\psi_i^{2j-1}\} = \\
&\{y \in \mathbb{X} \ \mid \ y = r^{2j-1} + \sum_{i=1}^{n} (\sigma_i^{2j-1}\gamma_i - \sigma_i^{2j-1}\psi_i^{2j-1}) z_i^{2j-1} \, , \\
&\qquad (\, \sigma_i^{2j-1}\gamma_i - \sigma_i^{2j-1}\psi_i^{2j-1}) \geq 0\} = \\
&\ r^{2j-1} + \{y \in \mathbb{X} \mid y = \sum_{i=1}^{n} \tau_i z_i^{2j-1} \, , \, \tau_i \in \Re^{+} \cup \{0\}\} \qquad \square
\end{aligned}$$

We now consider the problem of different locations of $\mathbf{P}_{\mathbb{S}}(x)$ with respect to $\mathbb{H}$. Clearly if $\mathbf{P}_{\mathbb{S}}(x)$ happens to be in $\mathbb{H}$ there is nothing to worry about, since, because of the fact that the metric projection operator is *idempotent* in this case we have $\mathbf{P}_{\mathbb{H}}(x) = \mathbf{P}_{\mathbb{S}}(x)$. If $\mathbf{P}_{\mathbb{S}}(x) \notin \mathbb{H}$ we may distinguish two different situations, namely where $\mathbf{P}_{\mathbb{S}}(x)$ is in the periphery of an extreme point $r^j \in \mathbb{E}$, and where this is not the case. We can make this intuitive idea precise in the following way:

Definition 2.5 *The translated polar cone of the translated conical hull $\mathbb{K}^j$ as defined in definition 2.4 is the following convex subset of $\mathbb{S}$:*

$$(\mathbb{K}^j)_0 = \{s \in \mathbb{S} \mid \langle (s - r^j), z^j \rangle \leq 0 \, \forall \, z^j \in con(\mathbb{Z}^j)\} \tag{9}$$

$\mathbf{P}_{\mathbb{S}}(x)$ *is in the periphery of the extreme point $r^j \in \mathbb{E}$ if* $\mathbf{P}_{\mathbb{S}}(x) \in (\mathbb{K}^j)_0$

The justification for this terminology will follow from the computation of $\mathbf{P}_{\mathbb{H}}(x)$ for this situation, which is presented in the next section. So concerning the positioning of $\mathbf{P}_{\mathbb{S}}(x)$ with respect to $\mathbb{H}$ we are left with the situation $\mathbf{P}_{\mathbb{S}}(x) \in \mathbb{D}$, where the set $\mathbb{D}$ is defined by:

$$\mathbb{D} = \mathbb{S} \setminus ((\bigcup_{j=1}^{2^n} (\mathbb{K}^j)_0) \cup \mathbb{H}) \tag{10}$$

The set $\mathbb{D}$ is *disconnected* - see J. Lee [6] - with convex components. So $\mathbf{P}_{\mathbb{S}}(x) \in \mathbb{D}$ means $\mathbf{P}_{\mathbb{S}}(x)$ in one of the components of $\mathbb{D}$. Of importance in relation to the projection onto $\mathbb{H}$ is of course the fact which part of $\mathbb{H}$ with respect to one of its decomposing terms $\mathbb{K}^j$ $(j \in \mathbb{J})$ $\mathbf{P}_{\mathbb{S}}(x)$ is facing when it is in one of the components of $\mathbb{D}$. To describe this situation we return to the adjacent extreme points of definition 2.3:

Definition 2.6 *Let $j \in \mathbb{J}$. Then the collection of n adjacent extreme points of $r^j \in \mathbb{E}$ is denoted by $\mathbb{E}^{r^j}$, where*

$$\mathbb{E}^{r^j} = \{r^k \in \mathbb{E} \mid r^k \textit{ is an adjacent extreme point of } r^j\}$$

The extremal subset of $\mathbb{H}$ associated with $r^j \in \mathbb{E}$ and one of its adjacent extreme points $r^k \in \mathbb{E}^{r^j}$ is given by:

$$\mathbb{E}^{r^j}_{r^k} = \{h \in \mathbb{H} \mid \ h = \lambda r^j + (1-\lambda) r^k \,,\ 0 \leq \lambda \leq 1\,\}$$

In the sequel we often need to consider $(n-1)$ of the adjacent extreme points of an extreme point r^j. There are n possibilities in this connection, and each choice is indicated by the parameter $i \in \mathbb{I}$ in the following way:

$$\mathbb{C}^j_i = \{r^{j_{i_l}} \in \mathbb{E}^{r^j} \mid l = 1, \ldots, n-1\} \tag{11}$$

If $\mathbf{P}_{\mathbb{S}}(x) \in \mathbb{D}$ then for some $j \in \mathbb{J}$ and for some $i \in \mathbb{I}$ the metric projection $\mathbf{P}_{\mathbb{S}}(x)$ looks out at a subset of $\mathbb{K}^j$ defined by:

Definition 2.7 *Given the cartesian product index set $\mathbb{I} \times \mathbb{J}$, and let (i,j) be an element of this index set. The boundary set of $\mathbb{H}$ associated with the pair (i,j) is the the following subset of $\mathbb{K}^j$:*

$$\mathbb{M}^j_i = \{(\mathbb{E}^{r^j}_{r^{j_{i_1}}} + \cdots + \mathbb{E}^{r^j}_{r^{j_{i_{n-1}}}}) \setminus ((\bigcup_{l=1}^{n-1} \{r^{j_{i_l}}\}) + \{r^j\}) \mid r^{j_{i_l}} \in \mathbb{C}^j_i\} \tag{12}$$

The limit points in $\mathbb{S}$ of $\mathbb{M}^j_i$ that are not in $\mathbb{M}^j_i$ - see J. Lee [6] are given by:

$$\mathbb{L}^j_i = \mathbb{C}^j_i \cup \{r^j\} \tag{13}$$

i.e. the interpretation of $\mathbb{C}^j_i$ from definition 2.6 is that it is the selection of $(n-1)$ of the n adjacent extreme points of r^j such that the remaining adjacent point $r^{j_{i_n}}$ of r^j in $\mathbb{E}^{r^j} \setminus \mathbb{C}^j_i$ is not a limit point of $\mathbb{M}^j_i$, or equivalently such that for the extremal subset associated with r^j and $r^{j_{i_n}}$ we have $\mathbb{E}^{r^j}_{r^{j_{i_n}}} \not\subset \mathbb{M}^j_i$.

The usefulness of the sets $\{\mathbb{M}^j_i \mid j \in \mathbb{J}\,,\, i \in \mathbb{I}\,\}$ is stressed by the following result:

Proposition 2.3

(1) Let $2j-1\,,\, 2j \in \mathbb{J}$. Then

$$\mathbb{M}^{2j-1}_i \subset \mathbb{K}^{2j} \quad \forall i \in \mathbb{I}$$

where $\{r^{2j-1}, r^{2j}\}$ is a pair of opposite extreme points of $\mathbb{H}$.

(2) Let $j \ \in \mathbb{J}$ *and* $i \in \mathbb{I}$*, then for any* $l \in \mathbb{J}$ *such that* $r^l \in \mathbb{E}^{r^j} \setminus \mathbb{C}_i^j$ *we have* $r^l \notin \mathbb{L}_i^j$

(3) Consider the pair of boundary sets $\{\mathbb{M}_l^{2j}, \mathbb{M}_m^k\}$*. Then*

$$r^k \in \mathbb{L}_l^{2j-1} \ \wedge \ \{r^{2j}, r^{2j-1}\} \textit{ opposite extreme points } \Leftrightarrow l = m$$

(4) Let $j \, , \, k \in \mathbb{J}$ *and* $i \, , \, m \in \mathbb{I}$*, then*

$$(r^j \in \mathbb{L}_i^k \wedge r^k \in \mathbb{L}_m^j) \Leftrightarrow \mathbb{M}_i^k = \mathbb{M}_m^j \ \wedge \ i = m$$

(5) Consider the boundary set $\mathbb{M}_i^j$*. Then*

$$\mathbb{M}_i^j = \mathbb{M}_i^{j_{i_l}} \ \forall \ j_{i_l} \in \mathbb{J} \textit{ such that } r^{j_{i_l}} \in \mathbb{C}_i^j$$

(6) Let $j \in \mathbb{J}$ *and* $i \in \mathbb{I}$*, and let* $\mathbb{M}_i^j$ *be given by equation* (12)*,* $\mathbb{E}_{r^{j_{i_n}}}^{r^j}$ *as in definition 2.6, with the indices* $i_n \, , \, j_{i_n}$ *such that* $r^{j_{i_n}} \in \mathbb{E}^{r^j} \setminus \mathbb{C}_i^j$

$$\begin{aligned} \mathbb{M}_i^j \quad &\subset \quad r^j + span(\{z_{i_1}^j, \ldots, z_{i_{n-1}}^j\}) \subset \mathbb{S} \qquad (14) \\ \mathbb{E}_{r^{j_{i_n}}}^{r^j} \quad &= \quad \{r^j + \lambda \sigma_{i_n}^j \psi_{i_n}^j z_{i_n}^j \mid 0 \le \lambda \le 1\} \end{aligned}$$

Proof: The results follow from definitions 2.3, 2.6, and 2.7. $\square$
Equation(14) reveals another interpretation of our choice parameter i in addition to the interpretation given in definition 2.7, namely it controls which one of the $(n-1)$-dimensional subspaces of $\mathbb{S}$ contains, when translated by the extreme point r^j the boundary set $\mathbb{M}_i^j$.
The notation used in equation (14) *in relation with the choice parameter* i *will be used consistently throughout this paper.*

This concludes our geometric description of the environment of $\mathbb{H}$ in our best approximation contekst. In the next section we investigate the best approximation from the finitely generated components of $\mathbb{H}$.

3 Explicit Computation of the Metric Projection onto a Translated Convex Cone

The first step is finding the best approximation to $x \in \mathbb{X}$ from $\mathbb{S}$, which, according to theorem 2.2 is unique. To this end the following set is introduced:

$$\mathbb{B} = \{b_i \in \mathbb{X} \mid i = 1, \ldots, n\} \qquad (15)$$

where $\mathbb{B}$ is an orthonormal basis for $\mathbb{S}$, i.e. $\mathbb{S} = span(\mathbb{Z}^j) = span(\mathbb{B})$ $(j \in \mathbb{J})$, and $\langle b_i, b_j \rangle = 1$ if $i = j$ and $\langle b_i, b_j \rangle = 0$ otherwise. In particular $\mathbb{B}$ can be constructed from $\mathbb{Z}^j$ through the *Gram-Schmidt orthogonalization process* - see e.g. F. Deutsch [2]. The best approximation to x from $\mathbb{S}$ is the Fourier expansion of $\mathbf{P}_{\mathbb{S}}(x)$ relative to $\mathbb{B}$:

$$\mathbf{P}_{\mathbb{S}}(x) = \sum_{i=1}^{n} \langle b_i, x \rangle b_i \tag{16}$$

Thus we commute in $\mathbb{S}$ between the orthonormal basis $\mathbb{B}$ in which calculations are easy using the Fourier-coefficients as coordinates, and the 'natural' basis $\mathbb{Z}^j$ for some $j \in \mathbb{J}$in which the results can be interpreted directly. Writing the expansion of $\mathbf{P}_{\mathbb{S}}(x)$ with respect to the basis $\mathbb{Z}^j$ as

$$\mathbf{P}_{\mathbb{S}}(x) = \sum_{i=1}^{n} \sigma_i^j \delta_i z_i^j \quad (\, \delta_i \in \Re \,, \; j \in \mathbb{J} \,) \tag{17}$$

with σ_i^j given in equation (8), what is sought is the relation between $\{\sigma_1^j \delta_1, \ldots, \sigma_n^j \delta_n\}$ and $\{\langle b_1, x \rangle, \ldots, \langle b_n, x \rangle\}$. This question is addressed by P. Halmos [3] and his development is followed here:

Consider the linear transformation $\mathbf{T}^j : \mathbb{S} \rightarrow \mathbb{S}$ defined by $\mathbf{T}^j z_i^j = b_i$. The matrix of this transformation with respect to the basis $\mathbb{Z}^j$ is denoted by (t_{ik}^j), i.e. $b_k = \mathbf{T}^j z_k^j = \sum_{i=1}^{n} t_{ik}^j z_i^j$. It follows that:

$$\begin{aligned} (t_{ik}^j) &= \mathbf{G}^{-1}(z_1^j, z_2^j, \ldots, z_n^j)(u_{ik}^j) \\ (u_{ik}^j) &= (\langle b_k, z_i^j \rangle) \end{aligned} \tag{18}$$

where $\mathbf{G}(z_1^j, z_2^j, \ldots, z_n^j)$ is the Gram matrix defined by

$$(\mathbf{G}(z_1^j, \ldots, z_n^j))_{i,k} = \langle \, z_k^j, z_i^j \, \rangle \quad (\, i, k = 1, \ldots, n \,) \tag{19}$$

Recalling that $\mathbf{G}(z_1^j, z_2^j, \ldots, z_n^j)$ - [2] - and (t_{ik}^j) - [3] - are invertible, it follows that (u_{ik}^j) is invertible. The relation between the sets of coordinates of $\mathbf{P}_{\mathbb{S}}(x)$ in respectively equation (16) and (17) follows from a direct computation:

$$\sigma_i^j \delta_i = \sum_{k=1}^{n} t_{ik}^j \langle b_k, x \rangle \tag{20}$$

Note that equation (20) is equivalent to solving the *normal equations* - see F. Deutsch [2]

Likewise using the inverse transformation $\mathbf{V}^j : \mathbb{S} \rightarrow \mathbb{S}$ defined by $z_k^j = \mathbf{V}^j b_k$ and with the matrix of this transformation with respect to $\mathbb{B}$ denoted by (v_{ik}^j), i.e. $z_k^j = \mathbf{V}^j b_k = \sum_{i=1}^{n} v_{ik}^j b_i$ where

$$(v_{ik}^j) = (u_{ik}^j)^{\,t} \tag{21}$$

where the superscript t denotes *transposition*, and so the Fourier coefficients of $\mathbf{P}_{\mathbb{S}}(x)$ in equation (16) as function of the coefficients in equation (17) is given by

$$\langle b_i, x\rangle = \sum_{k=1}^{n} v_{ik}^j \sigma_k^j \delta_k \tag{22}$$

We are now ready to investigate the projection of $\mathbf{P}_{\mathbb{S}}(x)$ onto the components of $\mathbb{H}$. We start with a more geometrically flavoured, equivalent formulation of the best approximation characterization theorem 2.1. To this end, we need to introduce a new, important geometric object first:

Definition 3.1 *Let $\mathbb{A}$ be a nonempty subset of $\mathbb{X}$. The dual cone of $\mathbb{A}$ is the set*

$$\mathbb{A}^0 = \{\, x \in \mathbb{X} \mid \langle x, a\rangle \leq 0 \quad \forall a \in \mathbb{A}\,\}$$

Theorem 3.1 *Let $h \in \mathbb{H}$. Then*

$h = \mathbf{P}_{\mathbb{H}}(x) \Leftrightarrow x - h \in (\mathbb{H} - h)^0$ □

For a proof of this theorem, see [2] or [5]. We can now give the following sharpening of theorem 2.1:

Proposition 3.1 *Let $j \in \mathbb{J}$, and let $y_0 \in \mathbb{K}^j$. Then*

$y_0 = \mathbf{P}_{\mathbb{K}^j}(\mathbf{P}_{\mathbb{S}}(x)) \Leftrightarrow \langle \mathbf{P}_{\mathbb{S}}(x) - y_0, z\rangle \leq 0 \,\forall\, z \in con(\mathbb{Z}^j) \wedge \langle \mathbf{P}_{\mathbb{S}}(x) - y_0, y_0 - r^j\rangle = 0$

Proof: $y_0 = \mathbf{P}_{\mathbb{K}^j}(\mathbf{P}_{\mathbb{S}}(x)) \Leftrightarrow (\mathbf{P}_{\mathbb{S}}(x) - y_0) \in (\mathbb{K}^j - y_0)^0 \Leftrightarrow$ $\langle \mathbf{P}_{\mathbb{S}}(x) - y_0, z^j - (y_0 - r^j)\rangle \leq 0 \,\forall\, z^j \in con(\mathbb{Z}^j)$. The non-positivity condition on the inner product holds for all $z^j \in con(\mathbb{Z}^j)$, and so in particular for the following two choices:

$$\begin{aligned} z^j = 0 \quad &\Rightarrow \quad \langle \mathbf{P}_{\mathbb{S}}(x) - y_0, y_0 - r^j\rangle \geq 0 \\ z^j = 2(y_0 - r^j) \quad &\Rightarrow \quad \langle \mathbf{P}_{\mathbb{S}}(x) - y_0, y_0 - r^j\rangle \leq 0 \end{aligned}$$ □

Now, with reference to the discussion in the previous section, $\mathbf{P}_{\mathbb{S}}(x)$ is in the periphery of some extreme point $r^j \in \mathbb{E}$, then we have the following useful corollary to proposition 3.1:

Corollary 3.1 *Let $\mathbf{P}_{\mathbb{S}}(x) \in (\mathbb{K}^j)_0$ for some $j \in \mathbb{J}$.*
Then $\mathbf{P}_{\mathbb{K}^j}(\mathbf{P}_{\mathbb{S}}(x)) = r^j$.

Proof: Substitute r^j for y_0 in proposition 3.1. □

In the next section, as a lemma to our main result, we give easily computable necessary and sufficient conditions for $\mathbf{P}_{\mathbb{S}}(x)$ to be in $(\mathbb{K}^j)_0$.

Before investigating the situation in which $\mathbf{P}_{\mathbb{S}}(x)$ is not in $\mathbb{H}$, nor in one of the translated polar cones, we digress a little by presenting some additional results in which polar, translated cones are compared with dual cones, and a few results are presented concerning best approximations from dual cones. Our motivation is on the one hand because it provides some credit for the here introduced translated polar cones, and on the other had it sheds some light on the, for computational purposes less tractable dual cones.

Theorem 3.2 ***(Translated Polar - and Dual Cones)*** *Let $j \in \mathbb{J}$, and let $\mathbb{N}_\epsilon$ be any ϵ-environment of $0 \in \mathbb{S}$. Then*

(1)

$$(r^j + con(\mathbb{Z}^j))^0 = (r^j + con(\mathbb{Z}^j))_0 \Leftrightarrow r^j = 0$$

(2)

$$(r^j + con(\mathbb{Z}^j))_0 \neq \emptyset$$

(3)

$$\mathbb{N}_\epsilon \subset \mathbb{H} \Rightarrow \mathbb{H}^0 = \emptyset \Rightarrow (r^j + con(\mathbb{Z}^j))^0 = \emptyset$$

(4)

$$\mathbb{N}_\epsilon \subset (r^j + con(\mathbb{Z}^j)) \Rightarrow (r^j + con(\mathbb{Z}^j))^0 = \emptyset$$

(5) Assume $\mathbb{N}_\epsilon \not\subset \mathbb{H}$, and let $r^k \in \mathbb{E}$ be such that $\left\|r^k\right\| < \left\|r^j\right\| \forall\, r^j \in \mathbb{E}$, $(j \neq k)$. Consider the set

$$\mathbb{F} = \mathbb{H}^0 \setminus (\bigcup_{i=1}^{n}((r^{k_i} + con(\mathbb{Z}^{k_i}))_0 \mid r^{k_i} \in \mathbb{E}^{r^k}) \cup (r^k + con(\mathbb{Z}^k))_0)$$

Then

$$s \in \mathbb{F} \Rightarrow \mathbf{P}_{\mathbb{H}}(s) \in \mathbb{M}_i^k$$

(6)

$$s \in \mathbb{H}^0 \setminus \mathbb{F} \Rightarrow \mathbf{P}_{\mathbb{H}}(s) \begin{cases} = r^k \ \textit{if}\, s \in (r^k + con(\mathbb{Z}^k))_0 \\ = r^{k_i}\ ,\ r^{k_i} \in \mathbb{E}^{r^k}\ \textit{otherwise} \end{cases}$$

Proof:

(5) The set $\mathbb{D}$ is *disconnected* - see J. Lee [6] - with convex components. If s is in one of these components it follows from our construction that the point of $\mathbb{H}$ closest to s is an element of the sum of extremal subsets of $\mathbb{H}$, but without the extreme points, it faces.

(6) If $s \notin (r^k + con(\mathbb{Z}^k))_0$ then $s \in \bigcup_{i=1}^n((r^{k_i} + con(\mathbb{Z}^{k_i}))_0 \mid r^{k_i} \in \mathbb{E}^{r^k})$. This set is disconnected with components $(r^{k_i} + con(\mathbb{Z}^{k_i}))_0$. Hence s must be in one of these components. The result now follows from corollary 3.1. $\square$

Let us pick up our main story line again and assume that $\mathbf{P}_{\mathbb{S}}(x)$ is in some component of the set $\mathbb{D}$ given in equation (10). This means that for some $j \in \mathbb{J}$ and for some $i \in \mathbb{I}$ the metric projection $\mathbf{P}_{\mathbb{S}}(x)$ faces the set $\mathbb{M}_i^j \subset \mathbb{K}^j$. Before presenting the result for computing $\mathbf{P}_{\mathbb{K}^j}(\mathbf{P}_{\mathbb{S}}(x))$ with $\mathbf{P}_{\mathbb{S}}(x) \in \mathbb{D}$ we need to introduce the following subset of $\Re^n$:

$$\Xi^j = \overbrace{\left\{ \begin{pmatrix} \langle b_1, \mathbf{P}_{\mathbb{S}}(x) - r^j \rangle \\ \cdots \\ \cdots \\ \langle b_{n-1}, \mathbf{P}_{\mathbb{S}}(x) - r^j \rangle \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 \\ \cdot \\ \cdot \\ 0 \\ \langle b_n, \mathbf{P}_{\mathbb{S}}(x) - r^j \rangle \end{pmatrix} \right\}}^{\sum_{i=1}^{n-1} \binom{n}{i} = 2^n - 2 \text{ elements}} \tag{23}$$

where the vectors of this set are composed from all possible choices of elements from the two $\Re^n$ vectors $(\langle \mathbf{P}_{\mathbb{S}}(x) - r^j, b_i \rangle)$ and 0 with *at least one element from each*, and where $\mathbf{P}_{\mathbb{S}}(x)$ is given in equations (16) and (17), $b_i \in \mathbb{B}$ - cf. equation (15).

Theorem 3.3 *Suppose* $\mathbf{P}_{\mathbb{S}}(x) \in \mathbb{D}$, *i.e.* $\exists\ j \in \mathbb{J}$, $i \in \mathbb{I}$ *such that the metric projection* $\mathbf{P}_{\mathbb{S}}(x)$ *faces the set* $\mathbb{M}_i^j \subset \mathbb{K}^j$. *Then there exists a unique element* $\xi_0^j \in \Xi^j$ *such that*

$$\mathbf{P}_{\mathbb{K}^j}(\mathbf{P}_{\mathbb{S}}(x)) = \sum_{i=1}^{n} \sigma_i^j \rho_{0_i} z_i^j + r^j$$

where $(\sigma^j \rho_0) = ((\sigma_i^j \rho_{0_i})) \in \Re^n$ *is given by*

$$(\sigma^j \rho_0) = (t_{ik}^j) \xi_0^j$$

with (t_{ik}^j) *the matrix of the base transformation* $\mathbf{T}^j : \mathbb{S} \to \mathbb{S}$ *given in equation (18). In particular the best approximation to* $\mathbf{P}_{\mathbb{S}}(x)$ *from* $\mathbb{K}^j$ *is found in at most* $(2^n - 2)$ *trials.*

Proof: It follows from proposition 2.1 that there is a unique projection onto $\mathbb{K}^j$. This solution y_0 can be represented in the following way:

$$y_0 = \sum_{i=1}^{n} \sigma_i^j \rho_{0_i} z_i^j + r^j = \sum_{i=1}^{n} \sigma_i^j (\rho_{0_i} + \psi_i^j) z_i^j \quad (\rho_{0_i} \geq 0) \tag{24}$$

According to the orthogonality condition of proposition 3.1 for y_0 to be a solution, the vectors $\mathbf{P}_{\mathbb{S}}(x) - y_0$ and $y_0 - r^j$ with representations with respect to the basis $\mathbb{Z}^j$ of $\mathbb{S}$ given by

$$\begin{aligned} \mathbf{P}_{\mathbb{S}}(x) - y_0 &= \sum_{i=1}^{n} \sigma_i^j (\delta_i - \rho_{0_i} - \psi_i^j) z_i^j \\ y_0 - r^j &= \sum_{i=1}^{n} \sigma_i^j \rho_{0_i} z_i^j \end{aligned} \tag{25}$$

must be orthogonal to each other. Using the base transformation $\mathbf{V}^j : \mathbb{S} \to \mathbb{S}$ the vectors $\mathbf{P}_{\mathbb{S}}(x) - y_0$ and $y_0 - r^j$ have the following equivalent representation with respect to the basis $\mathbb{B}$ of $\mathbb{S}$:

$$\begin{aligned}\mathbf{P}_{\mathbb{S}}(x) - y_0 &= \sum_{i=1}^{n} \langle b_i, \mathbf{P}_{\mathbb{S}}(x) - y_0 \rangle b_i \quad (\,\langle b_i, \mathbf{P}_{\mathbb{S}}(x) - y_0 \rangle = \sum_{j=1}^{n} v_{ik}^j \sigma_i^j (\delta_i - \rho_{0_i} - \psi_i^j)\,) \\ y_0 - r^j &= \sum_{i=1}^{n} \langle b_i, y_0 - r^j \rangle b_i \quad (\,\langle b_i, y_0 - r^j \rangle = \sum_{j=1}^{n} v_{ik}^j \sigma_i^j \rho_{0_j}\,) \qquad (26)\end{aligned}$$

$\mathbf{P}_{\mathbb{S}}(x) - y_0$ and $y_0 - r^j$ are orthogonal to each other if (and only if) $\mathbf{P}_{\mathbb{S}}(x) - y_0$ is in the subspace of $\mathbb{S}$ that is the orthogonal complement of the subspace of $\mathbb{S}$ of which $y_0 - r^j$ is an element. In other words, the Fourier-coefficients of the representation of $\mathbf{P}_{\mathbb{S}}(x) - y_0$ with respect to $\mathbb{B}$ that may be non-zero, *must* be zero in the representation of $y_0 - r^j$ with respect to $\mathbb{B}$, and the other way around. There are $\sum_{i=1}^{n-1} \binom{n}{i} = 2^n - 2$ ways, with $\binom{\cdot}{\cdot}$ the binomial coefficient, in which $\mathbb{S}$ can be split into orthogonal complements. To be specific assume that $y_0 - r^j \in span\{\{b_1, \ldots, b_m\}\} \wedge \mathbf{P}_{\mathbb{S}}(x) - y_0 \in span\{\{b_{m+1}, \ldots, b_n\}\}$ or equivalently $y_0 - r^j \notin span\{\{b_{m+1}, \ldots, b_n\}\} \wedge \mathbf{P}_{\mathbb{S}}(x) - y_0 \notin span\{\{b_1, \ldots, b_m\}\}$. This is in turn equivalent to the following system of linear equations in the unknown coefficients $\rho_0 = (\rho_{0_i})$ in the representation of y_0 with respect to $\mathbb{Z}^j$:

$$(v_{ik}^j)(\sigma^j \rho_0) = \begin{pmatrix} \langle b_1, \mathbf{P}_{\mathbb{S}}(x) - r^j \rangle \\ \cdots \\ \cdots \\ \langle b_m, \mathbf{P}_{\mathbb{S}}(x) - r^j \rangle \\ 0 \\ . \\ 0 \end{pmatrix} \qquad (27)$$

where the $n \times n$ matrix (v_{ik}^j) is defined in equation (21). Note that the right hand side of equation (27) is an element of the set Ξ^j defined in equation (23). Recalling that (v_{ik}^j), being the matrix of the base transformation $\mathbf{V}^j : \mathbb{S} \to \mathbb{S}$, is non-singular there is a unique solution ρ_0. Indeed,

$$(\sigma^j \rho_0) = (t_{ik}^j) \begin{pmatrix} \langle b_1, \mathbf{P}_{\mathbb{S}}(x) - r^j \rangle \\ \cdots \\ \cdots \\ \langle b_m, \mathbf{P}_{\mathbb{S}}(x) - r^j \rangle \\ 0 \\ . \\ 0 \end{pmatrix} \qquad (28)$$

with (t_{ik}^j) given in equation (18). Of course it should be checked that y_0 really is in $\mathbb{K}^j$:

$$y_0 \in \mathbb{K}^j \Leftrightarrow \rho_0 \geq 0 \qquad (29)$$

Inequalities for vectors are to be understood component wise. Suppose y_0 is in $\mathbb{K}^j$. According to proposition 3.1 for y_0 to be a solution subsequently the non-positivity test must be passed:

$$\begin{aligned} \langle \mathbf{P}_{\mathbb{S}}(x) - y_0, z^j \rangle \leq 0 \,\forall\, z^j \in con(\mathbb{Z}^j) \quad &\Leftrightarrow \\ \langle \mathbf{P}_{\mathbb{S}}(x) - y_0, z_i^j \rangle \leq 0 \,\forall\, i \in \mathbb{I} \quad &\Leftrightarrow \\ \mathbf{G}(z_1^j, \ldots, z_n^j)(\sigma^j(\delta - \rho_0 - \psi^j)) \leq 0 & \end{aligned} \tag{30}$$

It should be added here that the test (29) is redundant in the sense that y_0 satisfying the orthogonality condition of proposition 3.1 and passing the test (30) *is* an element of $\mathbb{K}^j$. The point is that ρ_0 needs to be available anyway for the test (30), and so if y_0 satisfying the orthogonality condition does *not* pass the test (29), the test (30) can be skipped, and the procedure is repeated by choosing another element from the set Ξ^j as right hand side of equation (27). Because the solution set Ξ^j is exhaustive, the existence of the solution guarantees that one of the elements of Ξ^j will satisfy both conditions of proposition 3.1; the uniqueness of the solution implies that the procedure can be terminated as soon as this element of Ξ^j is found. $\square$

Theorem 3.3 has actually the flavour of an *existence result* where the existence is proved by giving a formal construction for the solution. For, unless n is vey small, the solution in theorem 3.3 may not render to be vey useful in practice, since it may cost $2^n - 2$ trials to find the solution, *given the component* of $\mathbb{D}$ in which $\mathbf{P}_{\mathbb{S}}(x)$ is located. But this assumption can in general not be made, and then the computation of the best approximation adds up to at most $2^n(2^n - 2)$ trials. So the question naturally arises what would happen if we would know in which component of $\mathbb{D}$ the metric projection $\mathbf{P}_{\mathbb{S}}(x)$ is located, i.e. if we would know which particular set $\mathbb{M}_i^j$ - cf. equation (12) - $\mathbf{P}_{\mathbb{S}}(x)$ is facing. From the proof of theorem 3.3 it follows that two different translates of the sought best approximation y_0 have to be located respectively in the components of an orthogonal decomposition of $\mathbb{S}$, namely a translate with respect to $\mathbf{P}_{\mathbb{S}}(x)$ and another one with respect to the extreme point $r^j \in \mathbb{E}$ associated with the set $\mathbb{M}_i^j$. The next corollary to theorem 3.3 shows that this information gives us a lead for a direct computation of the best approximation.

Corollary 3.3 *Let $j \in \mathbb{J}$ and $i \in \mathbb{I}$, and suppose the metric projection of $x \in \mathbb{X}$ onto $\mathbb{S}$, $\mathbf{P}_{\mathbb{S}}(x)$ is facing the set $\mathbb{M}_i^j \subset \mathbb{K}^j$ - cf. defintion 2.7 - in one of the components of $\mathbb{D}$. Then*

$$\mathbf{P}_{\mathbb{K}^j}(\mathbf{P}_{\mathbb{S}}(x)) = \sum_{l=1}^{n} \sigma_{i_l}^j \rho_{0_{i_l}} z_{i_l}^j + r^j$$

where $(\sigma^j \rho_0) = ((\sigma_{i_l}^j \rho_{0_{i_l}}))\in \Re^n$ *is given by*

$$(\sigma^j \rho_0) = (t_{i_l i_k}^j) \begin{pmatrix} \langle b_{i_1}^j, \mathbf{P}_{\mathbb{S}}(x) - r^j \rangle \\ \cdots \\ \cdots \\ \langle b_{i_{n-1}}^j, \mathbf{P}_{\mathbb{S}}(x) - r^j \rangle \\ 0 \end{pmatrix}$$

and $\mathbb{B}^j_{i_n} = \{b^j_{i_1}, b^j_{i_2}, \dots, b^j_{i_n}\}$ *is the through Gram-Schmidt constructed orthonormal basis from* $\mathbb{Z}^j = \{z^j_{i_1}, \dots, z^j_{i_{n-1}}, z^j_{i_n}\}$ *such that* $b^j_{i_n} = z^j_{i_n} / \left\| z^j_{i_n} \right\|$, *and the index* $i_n \in \mathbb{I}$ *is such that* $r^{j_{i_n}} \in \mathbb{E}^{r^j} \setminus \mathbb{C}^j_i$, *i.e.*

$$\mathbb{E}^{r^j}_{r^{j_{i_n}}} = \{r^j + \lambda \sigma^j_{i_n} \psi^j_{i_n} z^j_{i_n} \mid 0 \le \lambda \le 1\} \not\subset \mathbb{M}^j_i$$

and finally

$$\begin{aligned}
(t^j_{i_l i_k}) &= \mathbf{G}^{-1}(z^j_{i_1}, z^j_{i_2}, \dots, z^j_{i_n})(u^j_{i_l i_k}) \\
(u^j_{i_l i_k}) &= (\langle z^j_{i_l}, b_{i_k} \rangle)
\end{aligned}$$

Proof: The proof is a direct consequence of theorem 3.3 combined with proposition 2.3, and definition 2.6 □

Corollary 3.3 shows that the best approximation onto a translated convex cone as one of the components of the set $\mathbb{H}$ can be computed *directly*, specifically without any searching once it is known in which component of $\mathbb{D}$ $\mathbf{P}_{\mathbb{S}}(x)$ is located. Hence for this result to be useful the possibility to establish the location of $\mathbf{P}_{\mathbb{S}}(x)$ with respect to $\mathbb{H}$ now becomes an urgent matter. And of course, this location determination should be established with as little computational burden as possible, in any case very much less than a compuation of the 'order of $2^n(2^n - 2)$ trials'. This important location determination problem is analysed in the next section.

4 Location Determination of a Point with respect to a Convex Set

In this section we investigate the problem of establishing the position of $\mathbf{P}_{\mathbb{S}}(x)$ with respect to $\mathbb{H}$. Despite the abundance of 'structural regularity'in our problem setting, this turns out to be an all but trivial problem. Indeed, we regard the results of this section as our main results. We present three theorems which give complete solutions to this problem using different objects to solve this problem.

More specifically theorem 4.1 below uses, apart from the characterization of the membership of a translated polar cone - see proposition 4.1 below - only *'distances and angles'* to establish the location of $\mathbf{P}_{\mathbb{S}}(x)$ with respect to $\mathbb{H}$. This fact makes this result worth noting, despite the fact that it may require some computational effort in actual applications. On the other hand theorems 4.2 and 4.3 below use instead of distances *coordinates* with respect to the basis $\mathbb{Z}^j$. Here an optimal use is made of the geometric structure of our problem setting. This solution requires lowest, possibly minimal with respect to any approach to solve the best approximation problem addressed here, computational effort. We give a short evaluation in this connection at the end of this section.

To start with the first solution mentioned above, let us first establish a computable solution for membership of a translated polar cone.

Proposition 4.1 *Let $j \in \mathbb{J}$, and let $\mathbf{P}_{\mathbb{S}}(x) = \sum_{i=1}^{n} \sigma_i^j \delta_i z_i^j$. Then*

$$\mathbf{P}_{\mathbb{S}}(x) \in (\mathbb{K}^j)_0 \Leftrightarrow \mathbf{G}(z_1^j, \ldots, z_n^j)(\sigma^j(\delta - \psi^j)) \leq 0$$

where $(\sigma^j(\delta - \psi^j)) = (\sigma_i^j(\delta_i - \psi_i^j))$

Proof: The proof follows directly from definition 2.5. $\square$

Lemma 4.1 *Let $j \in \mathbb{J}$, and let $\mathbf{P}_{\mathbb{S}}(x) \notin \mathbb{H}$. Then*

$$\mathbf{P}_{\mathbb{S}}(x) \in (\mathbb{K}^j)_0 \Leftrightarrow \left\|\mathbf{P}_{\mathbb{S}}(x) - r^j\right\| < \|\mathbf{P}_{\mathbb{S}}(x) - h\| \ \forall\, h \in \mathbb{H} \setminus \{r^j\}$$

Proof: This follows from the fact that for $h \in \mathbb{H} \setminus \{r^j\}$ we have $\angle((\mathbf{P}_{\mathbb{S}}(x) - r^j)(\mathbf{P}_{\mathbb{S}}(x) - h)) > \pi/2$. $\square$

Theorem 4.1 *Let $j \in \mathbb{J}$, $i \in \mathbb{I}$ and let $\mathbf{P}_{\mathbb{S}}(x) = \sum_{k=1}^{n} \sigma_k^j \delta_k z_k^j$. Then*

(1) $\mathbf{P}_{\mathbb{S}}(x) \in \mathbb{H} \Leftrightarrow \delta \in \Gamma$

(2) $\mathbf{P}_{\mathbb{S}}(x) \in (\mathbb{K}^j)_0 \Leftrightarrow$

$$\{[\left\|\mathbf{P}_{\mathbb{S}}(x) - r^j\right\| < \left\|\mathbf{P}_{\mathbb{S}}(x) - r^i\right\| \ \forall\, r^i \in \mathbb{E} \setminus \{r^j\}] \ \wedge$$

$$[\left\|\mathbf{P}_{\mathbb{S}}(x) - r^{j_i}\right\| < \left\|\mathbf{P}_{\mathbb{S}}(x) - r^k\right\| \ \forall\, r^{j_i} \in \mathbb{E}^{r^j}\,,$$
$$r^k \text{ the opposite extreme point of } r^j] \ \vee$$

$$[\left\|\mathbf{P}_{\mathbb{S}}(x) - r^{j_{i_n}}\right\| < \left\|\mathbf{P}_{\mathbb{S}}(x) - r^i\right\| \ \forall\, r^i \in \mathbb{E} \setminus \{r^{j_{i_n}}\}] \ \wedge$$

$$[\exists\, r^{j_{i_{n_{m_l}}}} \in \mathbb{E}^{r^{j_{i_n}}} \text{ such that } \left\|\mathbf{P}_{\mathbb{S}}(x) - r^{j_{k_n}}\right\| < \left\|\mathbf{P}_{\mathbb{S}}(x) - r^{j_{i_{n_{m_l}}}}\right\|\,,$$
$$r^{j_{k_n}} \text{ the opposite extreme point of } r^{j_{i_n}}] \ \wedge$$

$$[\left\|\mathbf{P}_{\mathbb{S}}(x) - r^{j_{i_n}}\right\| < \left\|\mathbf{P}_{\mathbb{S}}(x) - r^j\right\|] \ \wedge$$

$$[\left\|\mathbf{P}_{\mathbb{S}}(x) - r^j\right\| < \left\|\mathbf{P}_{\mathbb{S}}(x) - r^{j_{i_{n_{m_l}}}}\right\| \ \forall\, r^{j_{i_{n_{m_l}}}} \in \mathbb{E}^{r^{j_{i_n}}} \setminus \{r^j\}]\} \ \wedge$$

$$\{\mathbf{G}(z_1^j, \ldots, z_n^j)(\sigma^j(\delta - \psi^j)) \leq 0\}$$

(3) $\mathbf{P}_{\mathbb{S}}(x) \notin \mathbb{H} \ \wedge \ \mathbf{P}_{\mathbb{S}}(x) \notin (\mathbb{K}^j)_0 \ \forall j \in \mathbb{J} \ \wedge \ \mathbf{P}_{\mathbb{S}}(x)$ *is facing the set* $\mathbb{M}_i^j \subset \mathbb{K}^j$ *in one of the components of* $\mathbb{D} \Leftrightarrow$

$$\{[\left\|\mathbf{P}_{\mathbb{S}}(x) - r^j\right\| \leq \left\|\mathbf{P}_{\mathbb{S}}(x) - r^i\right\| \ \forall\, r^i \in \mathbb{E} \setminus \{r^j\}] \ \wedge$$

$$[\left\|\mathbf{P}_{\mathbb{S}}(x) - r^{j_i}\right\| < \left\|\mathbf{P}_{\mathbb{S}}(x) - r^k\right\| \ \forall\, r^{j_i} \in \mathbb{E}^{r^j}\,,$$

r^k *the opposite extreme point of* r^j] $\wedge$

[$r^{j_{i_n}} \in \mathbb{E}^{r^j}$ *is such that*

$$\langle(\mathbf{P}_{\mathbb{S}}(x) - r^j)(r^{j_{i_n}} - r^j)\rangle/(\left\|\mathbf{P}_{\mathbb{S}}(x) - r^j\right\| \left\|r^{j_{i_n}} - r^j\right\|) =$$
$$\min\{\langle(\mathbf{P}_{\mathbb{S}}(x) - r^j)(r^{j_{i_l}} - r^j)\rangle/(\left\|\mathbf{P}_{\mathbb{S}}(x) - r^j\right\| \left\|r^{j_{i_l}} - r^j\right\|)$$
$$\mid \forall\, r^{j_{i_l}} \in [E^{r^j}\}]\} \vee$$

$\{[\left\|\mathbf{P}_{\mathbb{S}}(x) - r^{j_{i_n}}\right\| \leq \left\|\mathbf{P}_{\mathbb{S}}(x) - r^i\right\| \,\forall\, r^i \in \mathbb{E} \setminus \{r^{j_{i_n}}\}]\ \wedge$

$[\exists\, r^{j_{i_{n_{m_l}}}} \in \mathbb{E}^{r^{j_{i_n}}}$ *such that* $\left\|\mathbf{P}_{\mathbb{S}}(x) - r^{j_{k_n}}\right\| < \left\|\mathbf{P}_{\mathbb{S}}(x) - r^{j_{i_{n_{m_l}}}}\right\|$,
$r^{j_{k_n}}$ *the opposite extreme point of* $r^{j_{i_n}}$] $\wedge$

$[\left\|\mathbf{P}_{\mathbb{S}}(x) - r^{j_{i_n}}\right\| < \left\|\mathbf{P}_{\mathbb{S}}(x) - r^j\right\|]\ \wedge$

$[\left\|\mathbf{P}_{\mathbb{S}}(x) - r^j\right\| \leq \left\|\mathbf{P}_{\mathbb{S}}(x) - r^{j_{i_{n_{m_l}}}}\right\| \,\forall\, r^{j_{i_{n_{m_l}}}} \in \mathbb{E}^{r^{j_{i_n}}} \setminus \{r^j\}]\}$

where the set $\mathbb{M}_i^j$ *is defined by*

$$\mathbb{M}_i^j = \{(\mathbb{E}^{r^j}_{r^{j_{i_1}}} + \cdots + \mathbb{E}^{r^j}_{r^{j_{i_{n-1}}}}) \setminus ((\bigcup_{l=1}^{n-1}\{r^{j_{i_l}}\}) + \{r^j\}) \mid r^{j_{i_l}} \in \mathbb{C}_i^j\}$$

Proof:

(1) This follows from equation (3).

(2) We discuss first the, what we call here 'non-degenerate case '- see below our description of the 'degenerate case ', that is we consider first the last condition in combination with the first possibility for the distance conditions. The interpretation of the shortest distance condition of $\mathbf{P}_{\mathbb{S}}(x)$ to the extreme point r^j and the condition on the distance of $\mathbf{P}_{\mathbb{S}}(x)$ to the extreme point r^k opposite to r^j reads as follows: let $r^{j_i} \in \mathbb{E}^{r^j}$ be the adjacent extreme point of r^j such that $\left\|\mathbf{P}_{\mathbb{S}}(x) - r^{j_i}\right\| < \left\|\mathbf{P}_{\mathbb{S}}(x) - r^{j_l}\right\| \,\forall\, r^{j_l} \in \mathbb{E}^{r^j} \setminus \{r^j, r^{j_i}\}$. Consider the 1-dimensional subspace $l_{r^k, r^{j_i}} = \{h \in \mathbb{H} \mid h = \lambda r^k + (1-\lambda) r^{j_i},\ \lambda \in \Re\}$, where r^k is the extreme point opposite to r^j. Note that $\{r^k, r^{j_i}\}$ is *not* a pair of opposite extreme points, and, for $n > 2$ they are also *not* adjacent with respect to one another. From the fact that r^k is opposite -, and r^{j_i} is adjacent to r^j, it follows - cf. definition 2.3 - that $r^k = \sum_{l=1}^n \sigma_l^k \psi_l^k z_l^k$, $r^{j_i} = \sum_{l=1}^n \sigma_l^k \psi_l^{j_i} z_l^k$, where $\psi_l^k = \psi_l^{j_i} \,\forall\, l \in \mathbb{I} \setminus \{l_m\} \wedge \psi_{l_m}^k \neq \psi_{l_m}^{j_i}$. The distance condition on the opposite extreme point r^k of r^j ensures that the product of the $z_{l_m}^k$ coordinates of $\mathbf{P}_{\mathbb{S}}(x)$ and r^j *relative* with respect to the 1-dimensional subspace $l_{r^k, r^{j_i}}$ is positive.

Now in the 'degenerate case ', where the set $\mathbb{H}$ has in one or more directions strong '*oblique and flat*' geometric features, it may happen that $\mathbf{P}_{\mathbb{S}}(x)$ is in the translated polar cone $(\mathbb{K}^j)_0$, although an adjacent point r^{j_i} of r^j is closer to $\mathbf{P}_{\mathbb{S}}(x)$ than r^j; this case follows from the previous one, basically by reversing the roles of $r^{j_i}$and r^j. The rest follows from proposition 4.1 and lemma 4.1.

(3) The distance conditions have basically the same interpretation as in the previous case, except that in the degenerate case - see case (2) above - the condition on the extreme opposite point fixes the component of $\mathbb{D}$ in which $\mathbf{P}_{\mathbb{S}}(x)$ is located, knowing that $\mathbf{P}_{\mathbb{S}}(x)$ is not in $\mathbb{H}$ nor in any of the translated polar cones. In the non-degenerate case the component of $\mathbb{D}$ in which $\mathbf{P}_{\mathbb{S}}(x)$ is located where $\mathbf{P}_{\mathbb{S}}(x)$ is facing the set $\mathbb{M}_i^j$ is determined completely by the extreme point r^j and $n-1$ of its adjacent extreme points $\mathbb{E}^{r^j}$. From the geometry of our problem setting it follows that

$$0 \leq \angle((\mathbf{P}_{\mathbb{S}}(x) - r^j), (r^{j_{i_l}} - r^j)) \leq \pi$$

and this angle is the largest for the adjacent extreme point of r^j, denoted by $r^{j_{i_n}}$ such that $\mathbb{E}^{r^j}_{r^{j_{i_n}}} \not\subset \mathbb{M}_i^j$. The result follows from the fact that the cosine is monotonically decreasing on $[0, \pi]$. $\square$

We start the second approach to the location determination problem mentioned in the introduction of this section by recalling equation (14) from proposition 2.3:

$$\mathbb{M}_i^j \subset r^j + span(\{z^j_{i_1}, \ldots, z^j_{i_{n-1}}\}) =: r^j + span(\mathbb{Z}^j \setminus \{z^j_{i_n}\}) \quad (i_n \in \mathbb{I}) \tag{31}$$

Stated in this way, equation (31) may be considered to be a defining equation for our choice parameter i.
The next proposition collects a number of useful properties of the sets $\mathbb{M}_i^j$. Together with proposition 2.3, these results show that the sets $\{\mathbb{M}_i^j\}$ are a good 'navigation system' for the boundary of $\mathbb{H}$ in $\mathbb{S}$.

Proposition 4.2

(1) Consider the pair of boundary sets $\{\mathbb{M}_m^{2j}, \mathbb{M}_m^k\}$. Then

$$\mathbb{M}_m^{2j} \cap \mathbb{M}_m^k = \emptyset \Leftrightarrow r^k \in \mathbb{L}_m^{2j-1} \ \wedge \ \{r^{2j}, r^{2j-1}\} \text{ opposite extreme points}$$

(2) Let $k \in \mathbb{I}$ and let $\{\mathbb{M}_{i^l}^{j^l} \mid l = 1, \ldots, k \ ; \ i^l \in \mathbb{I}, \, j^l \in \mathbb{J}\}$ be a collection of non-identical boundary sets such that there are no opposite pairs among the

collection of extreme points $\{r^{j^1}, \ldots, r^{j^k}\}$. *Then*

$$\bigcap_{l=1}^{k} \mathbb{M}_{i^l}^{j^l} \subset r^m + span(\mathbb{Z}^m \setminus \{z_{i_n^l}^m \mid l = 1, \ldots, k \,;\; i_n^l \in \mathbb{I},\, i_n^l \neq l_n^m \,,\, l \neq m\}),$$

$$m \in \mathbb{J} \text{ such that } r^m \in \bigcap_{l=1}^{k} \mathbb{L}_{i^l}^{j^l} \text{ and}$$

$$i_n^l \in \mathbb{I} \text{ such that } r^{j_{i_n^l}^l} \in \mathbb{E}^{r^{j^l}} \setminus \mathbb{C}_{i^l}^{j^l}$$

where we define $span(\emptyset) = 0$

Proof:

(1) This folows from equation (31) and proposition 2.3.

(2) This follows from (1) and proposition 2.3. □

Next we need to introduce sets which form a good navigation system for the complement of $\mathbb{H}$ in $\mathbb{S}$.

Definition 4.1 *Let* $j \in \mathbb{J}$ *and* $i \in \mathbb{I}$. *The border set of* $\mathbb{H}$ *with respect to the* i^{th}*-coordinate of the extreme point* $r^j = \sum_{k=1}^{n} \sigma_k^j \psi_k^j z_k^j$ *is the following subset of* $\mathbb{S}$*:*

$$\mathbb{S}_{\psi_i^j} = \{y = \sum_{k=1}^{n} \sigma_k^j \tau_k^j z_k^j \in \mathbb{S} \mid \sigma_i^j \tau_i^j < \sigma_i^j \psi_i^j\}$$

There are in total $2n$ *different border sets in accordance with the different values the parameter* ψ_i^j *can assume when the index pair* (i, j) *varies over the cartesian product index set* $\mathbb{I} \times \mathbb{J}$.
The half-closed versions of the border sets are defined as follows:

$$\bar{\mathbb{S}}_{\psi_i^j} = \{y = \sum_{k=1}^{n} \sigma_k^j \tau_k^j z_k^j \in \mathbb{S} \mid \sigma_i^j \tau_i^j \leq \sigma_i^j \psi_i^j\}$$

The importance of the border sets in our context is stipulated in the following result:

Proposition 4.3 *Let* $\mathbf{P}_{\mathbb{S}}(x) = \sum_{i=1}^{n} \sigma_i^m \delta_i z_i^m$ *for some* $m \in \mathbb{J}$. *Then*

$$\mathbf{P}_{\mathbb{S}}(x) \notin \mathbb{H} \Leftrightarrow \exists\, k \in \mathbb{I} \text{ such that } \forall\, l \in \{1, \ldots, k\} \text{ we have}$$

$$\sigma_{i^l}^{j^l} \delta_{i^l} < \sigma_{i^l}^{j^l} \psi_{i^l}^{j^l} \;(i^l \in \mathbb{I},\, j^l \in \mathbb{J}) \Leftrightarrow \mathbf{P}_{\mathbb{S}}(x) \in \bigcap_{l=1}^{k} \mathbb{S}_{\psi_{i^l}^{j^l}}$$

Proof: $\mathbf{P}_{\mathbb{S}}(x) \in \mathbb{H} \Leftrightarrow$ for any $j \in \mathbb{J}$ $\sigma_i^j \delta_i \geq \sigma_i^j \psi_i^j$ $\forall$ $i \in \mathbb{I}$ $\square$

The following result may be interpreted as the '*view*' of $\mathbf{P}_{\mathbb{S}}(x) \notin \mathbb{H}$ with respect to $\mathbb{H}$.

Proposition 4.4 *Let $k \in \mathbb{I}$ and let $\mathbf{P}_{\mathbb{S}}(x) \in \bigcap_{l=1}^{k} \mathbb{S}_{\psi_{i^l}^{j^l}}$. Then*

$$\bigcap_{l=1}^{k} \bar{\mathbb{S}}_{\psi_{i^l}^{j^l}} \cap \mathbb{H} = \bigcap_{l=1}^{k} \mathbb{M}_{\psi_{i^l}^{j^l}} \qquad \square$$

We are now ready to give our results concerning the location of $\mathbf{P}_{\mathbb{S}}(x) \notin \mathbb{H}$. Because of their different character, the results for the translated polar cones, and the components of $\mathbb{D}$ are presented in two separate theorems.

Theorem 4.2 *Let $j \in \mathbb{J}$, and let $\mathbf{P}_{\mathbb{S}}(x) = \sum_{i=1}^{n} \sigma_i^m \delta_i z_i^m$ for some $m \in \mathbb{J}$. Then*

$$\mathbf{P}_{\mathbb{S}}(x) \in (\mathbb{K}^j)_0 \Leftrightarrow \exists\, k \in \mathbb{I} \text{ such that } [\mathbf{P}_{\mathbb{S}}(x) \in \bigcap_{l=1}^{k} \mathbb{S}_{\psi_{i^l}^{j^l}} \wedge$$

$$\exists\, r^j = \sum_{l=1}^{n} \sigma_l^j \psi_l^j z_l^j \in \bigcap_{l=1}^{k} \mathbb{L}_{i^l}^{j^l} \text{ such that } \mathbf{G}(z_1^j, \ldots, z_n^j)(\sigma^j(\delta - \psi^j)) \leq 0]$$

Proof: The proof follows from theorem 4.2, and propositions 2.3, 4.1, 4.2, 4.3 and 4.4. According to lemma 4.1,
$\left\|\mathbf{P}_{\mathbb{S}}(x) - r^j\right\| < \|\mathbf{P}_{\mathbb{S}}(x) - r^m\|$ $\forall$ $r^m \in \bigcap_{l=1}^{k} \mathbb{L}_{i^l}^{j^l} \setminus \{r^j\}$. $\square$

Next we consider the situation where $\mathbf{P}_{\mathbb{S}}(x)$ is in one of the components of the set $\mathbb{D}$. We have to differentiate between two cases here depending on the fact either none of the coordinates δ_i - with respect to the basis $\mathbb{Y}$ - of $\mathbf{P}_{\mathbb{S}}(x)$ is between α_i and β_i, or at least one of them is.

Theorem 4.3 *Let $j \in \mathbb{J}$, and let $\mathbf{P}_{\mathbb{S}}(x) = \sum_{i=1}^{n} \sigma_i^m \delta_i z_i^m$ for some $m \in \mathbb{J}$. Then*

$\mathbf{P}_{\mathbb{S}}(x)$ is facing the set $\mathbb{M}_i^j \subset \mathbb{K}^j$ in one of the components of $\mathbb{D}$, where

$$\mathbb{M}_i^j = \{(\mathbb{E}_{r^{j_{i_1}}}^{r^j} + \cdots + \mathbb{E}_{r^{j_{i_{n-1}}}}^{r^j}) \setminus ((\bigcup_{l=1}^{n-1} \{r^{j_{i_l}}\}) + \{r^j\}) \mid r^{j_{i_l}} \in \mathbb{C}_i^j\} \iff$$

$$[\mathbf{P}_{\mathbb{S}}(x) \in \bigcap_{l=1}^{n} \mathbb{S}_{\psi^{j^l}_{i^l}} \ \wedge$$

$$\mathbf{P}_{\mathbb{S}}(x) \notin (\mathbb{K}^m)_0 \ \forall \ m \in \mathbb{J} \ \textit{such that} \ r^m \in \mathbb{E}^{r^j} \cup \{r^j\} \ , \ \{r^j\} = \bigcap_{l=1}^{n} \bar{\mathbb{S}}_{\psi^{j^l}_{i^l}} \cap \mathbb{H} \, ,$$

$$r^{j_{i_n}} \in \mathbb{E}^{r^j} \setminus \mathbb{C}^j_i \ \textit{such that} \ \langle (\mathbf{P}_{\mathbb{S}}(x) - r^j)(r^{j_{i_n}} - r^j) \rangle / (\left\| \mathbf{P}_{\mathbb{S}}(x) - r^j \right\| \left\| r^{j_{i_n}} - r^j \right\|)$$
$$= \min\{ \langle \, (\mathbf{P}_{\mathbb{S}}(x) - r^j)(r^{j_{i_l}} - r^j) \rangle / (\left\| \mathbf{P}_{\mathbb{S}}(x) - r^j \right\| \left\| r^{j_{i_l}} - r^j \right\|) \ | \ \forall r^{j_{i_l}} \in \mathbb{E}^{r^j} \}] \ \vee$$

$$[\mathbf{P}_{\mathbb{S}}(x) \in \bigcap_{l=1}^{k<n} \mathbb{S}_{\psi^{j^l}_{i^l}} \ \wedge$$

$$\mathbf{P}_{\mathbb{S}}(x) \notin (\mathbb{K}^m)_0 \ \forall \ m \in \mathbb{J} \ \textit{such that} \ r^m \in \bigcap_{i=1}^{l} \mathbb{L}^{j^l}_{i^l} \ \wedge$$

$$r^j \in \bigcap_{l=1}^{k} \mathbb{L}^{j^l}_{i^l} \ \wedge$$

$$r^{j_{i_n}} \in \mathbb{E}^{r^j} \ \textit{such that} \ r^{j_{i_n}} \notin \bigcap_{l=1}^{k} \mathbb{L}^{j^l}_{i^l}]$$

Proof: The proof follows from propositions 2.3, 4.2, 4.3 and 4.4. $\square$

Theorem 4.1 together with corollary 3.3 would imply that our method to calculate the best approximation from an arbitrary component of the set $\mathbb{H}$ would still be of *exponential order in n*. However, combining theorem 4.2 or theorem 4.3 with corollary 3.3 leads to a computation which is of *linear order in n* only. Clearly theorem 4.1 may be of theoretical value only, whereas theorems 4.2 and 4.3 may be of importance in particular in applications. Indeed, combining corollary 3.3 with theorem 4.3 leads to a further reduction in the computation of the best approximation, that we present here as a direct corollary to theorem 4.3.

Corollary 4.3 *Let* $k \in \{1, \ldots, n-1\}$, $\mathbf{P}_{\mathbb{S}}(x) \in \bigcap_{l=1}^{k} \mathbb{S}_{\psi^{j^l}_{i^l}}$, $\mathbf{P}_{\mathbb{S}}(x) \notin (\mathbb{K}^m)_0 \ \forall \ m \in \mathbb{J}$ *such that* $r^m \in \bigcap_{l=1}^{k} \mathbb{L}^{j^l}_{i^l}$*, and let* $j \in \mathbb{J}$ *be such that* $r^j \in \bigcap_{l=1}^{k} \mathbb{L}^{j^l}_{i^l}$.
Let $\mathbb{I}^k_n \subset \mathbb{I}$ *be such that* $\bigcap_{l=1}^{k} \bar{\mathbb{S}}_{\psi^{j^l}_{i^l}} \cap \mathbb{H} = \bigcap_{l=1}^{k} \mathbb{M}^{j^l_{i^l}} \subset r^j + span(\mathbb{Z}^j \setminus \{z^j_{i^l_n} \mid i^l_n \in \mathbb{I}^k_n\})$*, i.e.* $\mathbb{I}^k_n = \{i^1_n, \ldots i^k_n\}$, i^l_n *such that* $r^{j^l_{i^l_n}} \in \mathbb{E}^{r^{j^l}} \setminus \mathbb{C}^{j^l}_{i^l}$ *- cf. proposition 4.2(2).*

Then

$$\mathbf{P}_{\mathbb{K}^j}(\mathbf{P}_{\mathbb{S}}(x)) = \sum_{l=1}^{n} \sigma^j_{i_l} \rho_{0_{i_l}} z^j_{i_l} + r^j$$

where

$$
\begin{aligned}
\sigma^j_{i_l}\rho_{0_{i_l}} &= (t^j_{i_k i_m})_{i_l} \begin{pmatrix} \langle b^j_{i_1}, \mathbf{P}_{\mathbb{S}}(x) - r^j \rangle \\ \cdots \\ \cdots \\ \langle b^j_{i_{n-1}}, \mathbf{P}_{\mathbb{S}}(x) - r^j \rangle \\ 0 \end{pmatrix} \quad \textit{for} \quad i_l \in \mathbb{I} \setminus \mathbb{I}^k_n \\
\sigma^j_{i_l}\rho_{0_{i_l}} &= \sigma^j_{i_l}\psi^j_{i_l} \quad \textit{for} \quad i_l \in \mathbb{I}^k_n
\end{aligned}
$$

where $(t^j_{i_k i_m})_{i_l}$ *denotes the* i_l*-th row of the matrix* $(t^j_{i_k i_m})$ *defined in equation* (18)*, and* $\mathbb{B}^j_{i_n} = \{b^j_{i_1}, b^j_{i_2}, \ldots, b^j_{i_n}\}$ *is the through Gram-Schmidt constructed orthonormal basis from* $\mathbb{Z}^j = \{z^j_{i_1}, \ldots, z^j_{i_{n-1}}, z^j_{i_n}\}$ *such that* $b^j_{i_n} = z^j_{i_n} / \left\| z^j_{i_n} \right\|$*, and the index* $i_n \in \mathbb{I}^k_n$ $\square$

This completes the description of the metric projection onto the components of $\mathbb{H}$, as given in proposition 2.2. While the results of this - and the previous section may be of independent interest, our main goal is to compute the metric projection onto $\mathbb{H}$, and so the question which is still open is how to get from the metric projection onto the components of $\mathbb{H}$ to that onto $\mathbb{H}$ itself. On the other hand our intuition may tell us that we have already settled this problem, i.e. the metric project onto a component of $\mathbb{H}$ *is* the metric projection onto $\mathbb{H}$ itself. In the next question we show formally that our intuition is correct.

5 The Best Approximation from a Convex Set

The Boyle-Dykstra theorem establishes the convergence of an iterative procedure that computes best approximations from an intersection $\cap_{i=1}^n \mathbb{A}_i$ of a finite number of closed convex subsets $\mathbb{A}_i$ of a *Hilbert* space from the best approximations of the individual sets $\mathbb{A}_i$. A comprehensive treatment of this theorem can be found in the recent book of F. Deutsch [2]. The premises of the Boyle-Dykstra theorem correspond with the situation under consideration. Rather than giving the general formulation of this theorem, it is presented here in terms of the current situation. In order to do that, some straightforward notation has to be introduced first.

Let, for $n \in \mathfrak{N}$, $[n]$ denote ‘n modulo 2’, i.e.

$$[n] := \{1, 2\} \cap \{n - 2k \mid k = 0, 1, 2, \ldots\}$$

For any pair of opposite extreme points of $\mathbb{H}$ denote the associated translated conical hulls by $\mathbb{K}^{[1]}$ and $\mathbb{K}^{[2]}$ - cf. definition 2.4.

Theorem 5.1 ***(The Boyle-Dykstra theorem)***

Construct the following sequence $\{x_n\}$ *in* $\mathbb{S}$:

$$\begin{aligned} x_0 &= \mathbf{P}_{\mathbb{S}}(x), \quad e_{-1} = e_0 = 0, \\ x_n &= \mathbf{P}_{\mathbb{K}^{[n]}}(x_{n-1} + e_{n-2}) \\ e_n &= x_{n-1} + e_{n-2} - x_n \quad (n = 1, 2, \ldots) \end{aligned}$$

This sequence converges to the best approximation from $\mathbb{H}$ in the following way:

$$\lim_{n\to\infty} \|x_n - \mathbf{P}_{\mathbb{H}}(x)\| = 0$$

with $\|\cdot\|$ the norm on $\mathbb{X}$ □

For a proof of the Boyle-Dykstra theorem the excellent recent book of F. Deutsch [2] is recommended, in which also references to different applications of this theorem may be found. We are now ready to present the metric project onto the set $\mathbb{H}$.

Theorem 5.2 *Let $j \in \mathbb{J}$, $i \in \mathbb{I}$ and let $\mathbf{P}_{\mathbb{S}}(x) = \sum_{k=1}^{n} \sigma_k^j \delta_k z_k^j$. Then the* ***unique*** *best approximation to x from $\mathbb{H}$, $\mathbf{P}_{\mathbb{H}}(x)$, is given by one of the three following cases:*

(1) If $\mathbf{P}_{\mathbb{S}}(x) \in \mathbb{H}$, then $\mathbf{P}_{\mathbb{H}}(x) = \mathbf{P}_{\mathbb{S}}(x)$

(2) If $\mathbf{P}_{\mathbb{S}}(x) \in (\mathbb{K}^j)_0$, then $\mathbf{P}_{\mathbb{H}}(x) = r^j$

(3) If $\mathbf{P}_{\mathbb{S}}(x)$ is facing the set $\mathbb{M}_i^j \subset \mathbb{K}^j$ in one of the components of $\mathbb{D}$, then $\mathbf{P}_{\mathbb{H}}(x) = \mathbf{P}_{\mathbb{K}^j}(\mathbf{P}_{\mathbb{S}}(x))$

Proof:

(1) The claim follows from the fact that the projection operator $\mathbf{P}_{\mathbb{H}}$ is idempotent - see e.g. [2].

(2) $\mathbf{P}_{\mathbb{S}}(x) \in (\mathbb{K}^j)_0 \Rightarrow \mathbf{P}_{\mathbb{K}^j}(\mathbf{P}_{\mathbb{S}}(x)) = r^j$. Without loss of generality it may be assumed that j is odd, i.e. $j = 2k-1$ for some $k \in \{1, \ldots, 2^{n-1}\}$. But $r^j = r^{2k-1} \in \mathbb{K}^{2k}$ - cf. proposition 2.2, and $\mathbb{H} = \mathbb{K}^{2k-1} \cap \mathbb{K}^{2k}$. The result now follows from the Boyle-Dykstra theorem, because the sequence $\{x_n\}$ constructed in the theorem converges immediately to the constant sequence $\{r^j\}$, that is with reference to theorem 4.1 $x_i = r^j$, $e_{2i-1} = \mathbf{P}_{\mathbb{S}}(x) - r^j$, $e_{2i} = 0 \; \forall \, i \geq 1$

(3) Without loss of generality it may again be assumed that j is odd, i.e. $j = 2k-1$ for some $k \in \{1, \ldots, 2^{(n-1)}\}$. Using corollary **??** $\mathbf{P}_{\mathbb{K}^{2k-1}}(\mathbf{P}_{\mathbb{S}}(x)) \in \mathbb{M}_i^{2k-1} \subset \mathbb{K}^{2k-1}$ is computed. But $\mathbf{P}_{\mathbb{K}^{2k-1}}(\mathbf{P}_{\mathbb{S}}(x)) \in \mathbb{K}^{2k}$, since, by proposition 2.3, $\mathbb{M}_i^{2k-1} \subset \mathbb{K}^{2k}$. The result follows in the same way as in the previous case again from the Boyle-Dykstra theorem, that is the sequence $\{x_n\}$ constructed in the theorem converges immediately to the constant sequence $\{\mathbf{P}_{\mathbb{K}^{2k-1}}(\mathbf{P}_{\mathbb{S}}(x))\}$ □

The proof confirms the idea suggested at the end of section 4 that it only needs to be trivially confirmed that the metric projections onto the components of $\mathbb{H}$ *are* the ones on $\mathbb{H}$ itself.

Figure 1 below is meant to give a 'mental picture' to illustrate theorem 5.2, that is, although not a correct, rigorous representation, the figure does show the working of the theorem in an intuitively appealing way. Also note the 'pretty' geometric structure of our problem setting revealed by this figure. The idea of figure 1 is that it depicts a number of possible locations for $\mathbf{P}_{\mathbb{S}}(x)$ in the subspace $\mathbb{S}$, followed by the projection onto $\mathbb{H}$, which is suggested by the arrows in the figure, except of course for the situation where $\mathbf{P}_{\mathbb{S}}(x) \in \mathbb{H}$.

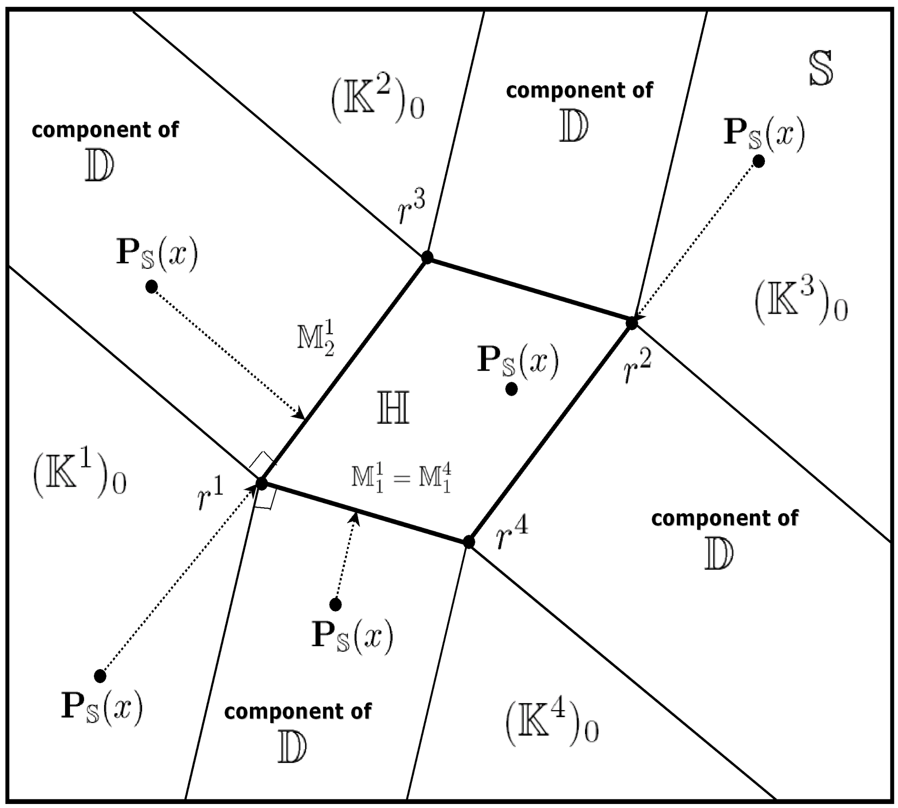

Figure 1: Illustration of metric projection onto $\mathbb{H}$

Theorem 5.2 gives a complete characterization of the best approximation to an element of an Inner Product Space from a convex subset in that space. In the next and final section we describe shortly a real-life application of this theorem.

6 Application: Production Allocation in Oil Industry

A common situation in production operations in oil industry is that a number of wells, say n wells, produce into a large piece of tubing called the *bulk header*, and from this header the common production is transported to the *bulk separator*, where the different phases, namely oil, water and gas, are separated, and the production rates of the separated phases are measured. The separated phases are transported further downstream.
Estimates, or rather predictions of the productions from the individual wells are obtained through a mathematical model for the production of the well. This model is established through a *well test.* A well test for, to be specific, well i of a group of wells is an experiment where the well is decoupled from the bulk header and connected to the *test header*, which is connected to the *test separator.* Here again the phase productions are measured, but this time those of well i only. - The phase productions from the well on test are recombined downstream from the test separator, and combined with the common bulk production from the other wells. - The well model is a mapping from quantities that drive the production, notably different types of pressures, to the phase-productions from the well. During the well test both the quantities in the domain of this mapping and those in its range are measured, and this information is used to establish this mapping from the measured data. When the well is back in production status the driving quantities are still available, and by processing these with the well model, predictions are obtained of the productions of the concerning well.
The idea of the production allocation problem is now to establish the contribution from the individual wells to the total production. Because the wells influence one another during production, the total production does in general not equal the sum of the separate predicted productions. In other words we have to establish the *best approximation* to the total prduction from the - predicted - individual well productions. In terms of the notation of the problem description in section 1 we have the following correspondence - we confine ourselves to considering one type of phase productions only:

$$
\begin{array}{rcl}
\textit{total production} & : & x \in \mathbb{X} \\
\textit{production of well } i & : & y_i \in \mathbb{Y} \\
\textit{number of wells} & : & n \\
\textit{admissible combinations} & : & \mathbb{H} \subset \mathbb{Y}
\end{array}
\tag{32}
$$

As for the 'admissible combinations', we note that negative contributions are not allowed. However somewhat counter intuitive may be the fact that the a contribution larger than one may be allowed. This means that the production from a certain well may be stimulated when it is producing in conjuction with one or more other wells. This may in particular occur in so-called multi-zone wells, where the individual wells are actually different zones in the reservoir that are visited by the production tubing of the multi-zone well. There are in general strong interactions between the different zones. On the other hand a multiple of the production because of interactions is generally highly unlikely, and will usually be assessed as ‘not admissible’.

Figure 2 below shows the total production from a mult-zone well, consisting of two zones A and B. let us agree to call the production from zone A y_1, and that from zone B y_2. The productions from the separate zones are calculated using their well models, and are predictions of their performance when the zones are producing separately.
It follows from figure 2 below that if zone B is going to contribute to the total production it will be at the expense of the performance of zone A when it produces alone; in another perspective it may be expected that zone A pushes away zone B almost completely when there are producing simultaneously.

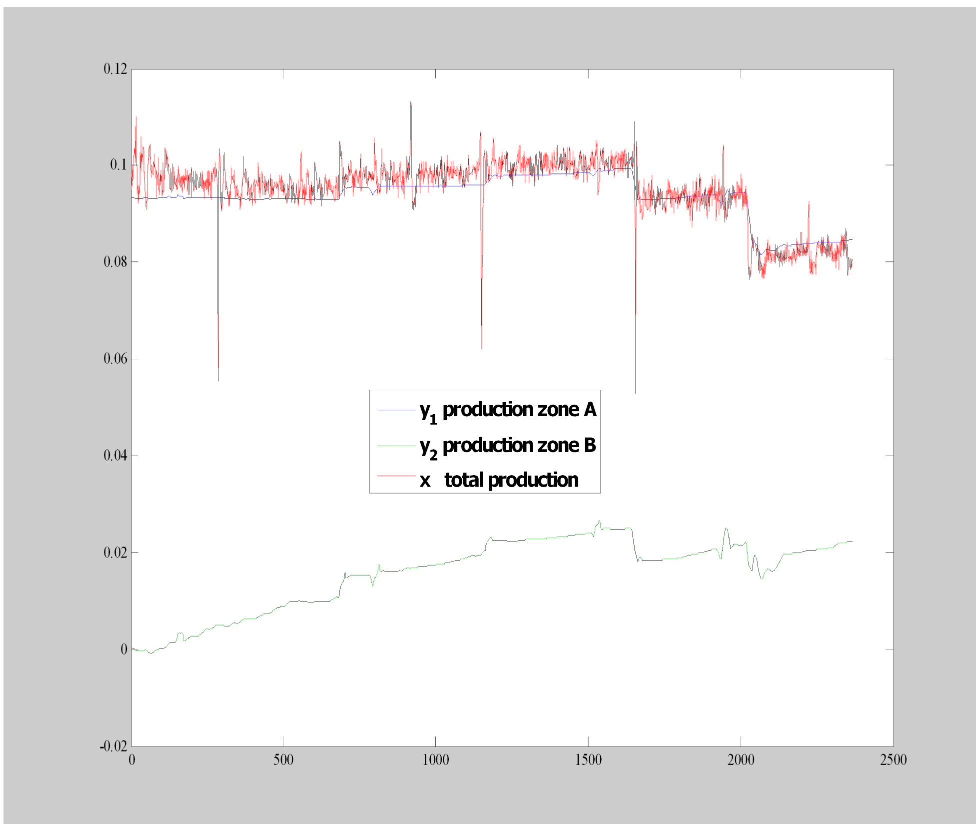

Figure 2: Total production x, and individual productions y_1 and y_2

When calculating $\mathbf{P}_{\mathbb{S}}(x)$ we find

$$\mathbf{P}_{\mathbb{S}}(x) = 1.05y_1 - 0.18y_2$$

The negative contribution from y_2 is of course physically impossible. A more complete assessment of this result is possible after defining the set $\mathbb{H}$ of admissible combinations. In production operations practice the position of $\mathbf{P}_{\mathbb{S}}(x)$ with respect to $\mathbb{H}$ may have *diagnostic* value in the sense that certain actions may be taken, or conclusions may be drawn on the basis of this 'relative 'position of $\mathbf{P}_{\mathbb{S}}(x)$. But for this it is imperative that a judicious choice is made of the admissible parameters via the 'shape 'parameters α and β in the defining equation (3) of $\mathbb{H}$. The table below shows the effect on $\mathbf{P}_{\mathbb{H}}(x)$ resulting from different choices for the shape parameters α and β. In order to appreciate the results presented here, we specify the representations of extreme points that have been used in the calculations:

$$\begin{aligned} r^1 &= \alpha_1 y_1 + \alpha_2 y_2 \\ r^2 &= \beta_1 y_1 + \beta_2 y_2 \\ r^3 &= \alpha_1 y_1 + \beta_2 y_2 \\ r^4 &= \beta_1 y_1 + \alpha_2 y_2 \end{aligned}$$

Exp num	$[\alpha_1; \alpha_2]$	$[\beta_1; \beta_2]$	view of $\mathbf{P}_{\mathbb{S}}(x)$
1	[.1 ; .05]	[2 ; 1]	$\mathbb{M}_2^1 = \mathbb{M}_2^3$
2	[.1 ; .1]	[.5 ; 1]	$(\mathbb{K}^2)_0$
3	[0 ; 0]	[.5 ; 3]	$\mathbb{M}_1^2 = \mathbb{M}_1^3$

Figure 3 below is a graphical representation of the different situations considered in the table.

Our choices of the shape parameters have admittedly been subordinate to a demonstration effect. In particular the choice made for the shape parameters in the third experiment is 'absurd'in the sense that the approximated total production is significantly larger than the measured total production. This is an illustration of such a 'not admissible'situation mentioned above when allowing the production rate contribution to be a multiple of the single contribution. In any case, these results nevertheless show the necessity for a sensible choice of the shape parameters. One way to do this is to cast the above problem in a totally different mathematical setting, specifically in terms of *ideal* theory as part of commutative algebra. The total production is then a member of the ideal generated by the separate productions, assuming polynomial representations for the well models. This leads to the following representation for the total production

$$x = \sum_{i=1}^{n} s_i y_i$$

where the s_i are polynomials, modeling the interactions between the wells. The s_i are subsequently used to construct the 'physically correct 'shape parameters.

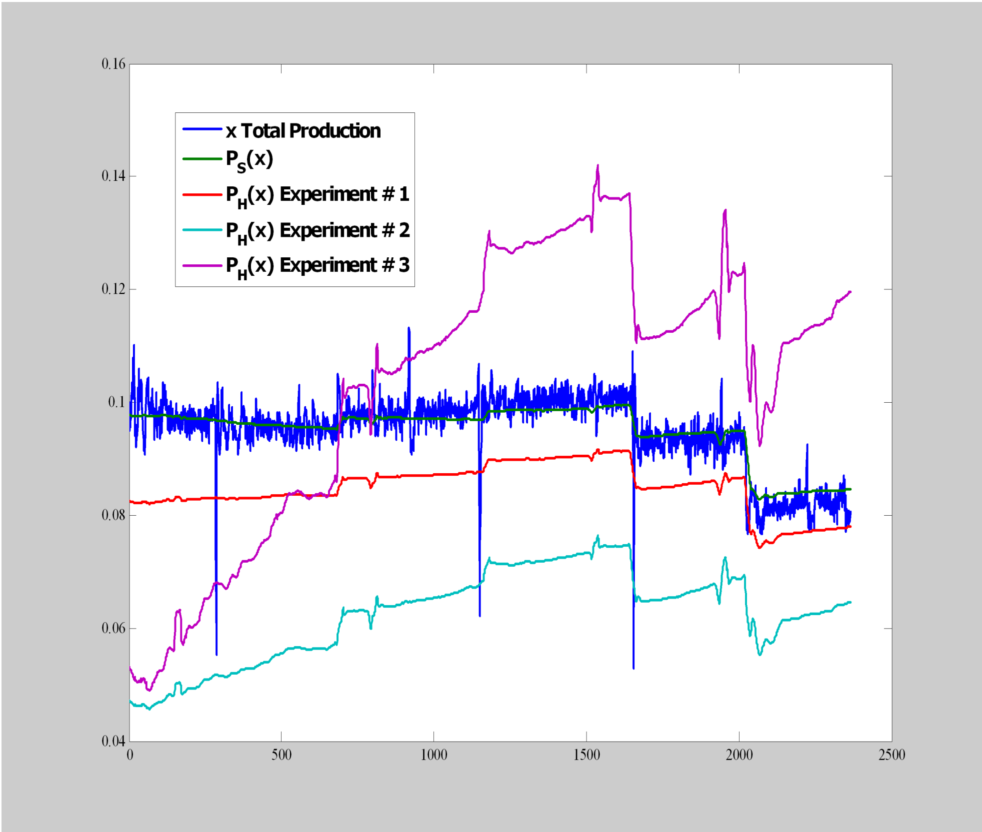

Figure 3: Total production x , $\mathbf{P}_{\mathbb{S}}(x)$, and $\mathbf{P}_{\mathbb{H}}(x)$ for various choices of the shape parameters

This new approach is beyond the scope of the present paper; first results in this direction may be found in [4].

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