Characterizations of Border Bases

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Abstract

This paper presents characterizations of border bases of zero-dimensional polynomial ideals that are analogous to the known characterizations of Gröbner bases. Based on a Border Division Algorithm, a variant of the usual Division Algorithm, we characterize border bases as border prebases with one of the following equivalent properties: special generation, generation of the border form ideal, confluence of the corresponding rewrite relation, reduction of S-polynomials to zero, and lifting of syzygies. The last characterization relies on a detailed study of the relative position of the border terms and their syzygy module. In particular, a border prebasis is a border basis if and only if all fundamental syzygies of neighboring border terms lift; these liftings are easy to compute.

Key words: border basis, division algorithm, syzygy module

1 Introduction

Auzinger and Stetter [1], Möller [5], and Mourrain [6], for instance, used border bases successfully to solve zero-dimensional polynomial systems of equations. One of the attractive features of border bases is that they behave numerically better than Gröbner bases. Their key role in numerical polynomial algebra is emphasized in, for example, the new book by Hans Stetter [7]. Recently, border bases have also been applied to statistics, cf. Caboara and Robbiano [2].

Kehrein, Kreuzer, and Robbiano [3] started to lay solid algebraic foundations of the theory of border bases. In the present paper we commence the brickwork and characterize border bases analogously to known characterizations

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of Gröbner bases; the latter are collected, for example, in Kreuzer and Robbiano [4, Theorem 2.4.1]. Our basic tool is the Border Division Algorithm, which we present in Section 2; it divides a polynomial by a list of polynomials. Unlike the usual Division Algorithm, it does not require a term ordering; instead, the divisor polynomials must constitute a border prebasis with respect to an order ideal of terms. Accordingly, the familiar reduction of leading terms is substituted by a reduction of terms with largest index, where the index measures a distance from the order ideal. This adapted reduction process is carefully designed to avoid infinite loops.

In Section 3, we apply the Border Division Algorithm and, thus, obtain immediately two characterizing properties of border bases: special generation of the ideal (see Proposition 9) and generation of the border form ideal (see Proposition 11). Another characterization of border bases uses the rewrite relation generated by the border prebasis. This is a trickier subject, because the rewrite relation is in general not Noetherian. In other words, we consider a reduction process with infinite loops (see Example 12). Surprisingly, confluence of this rewrite relation still is equivalent to the border basis property (see Proposition 14).

Section 4 presents a border basis analogue of Buchberger's criterion. Its proof uses Mourrain's characterization of border bases in terms of formal multiplication matrices (see Proposition 16) and a lengthy, but straightforward computation. This computation reveals that the border basis property is equivalent to the condition that the S-polynomials of neighboring border terms reduce to zero (see Proposition 18); here, two distinct border terms b and b' will be called neighbors, if $b = xb'$ or $xb = yb'$ for some indeterminates x, y.

The topic of the final section is the lifting of border syzygies. First, we study the relative position of the border terms and their syzygy module. In particular, we show that the border is connected with respect to the neighborhood relation (see Proposition 19) and that the neighbor syzygies generate the module of border syzygies (see Proposition 21). Next, we introduce the concept of a lifting of a border syzygy (see Definition 22) and show that, if liftings exist, then they will be computed easily for neighbor syzygies (see Example 23). Then, we characterize border bases as border prebases, for which all border syzygies lift or, equivalently, all neighbor syzygies lift. (see Proposition 25). After discussing possible border bases analogues of homogeneous syzygies, we conjecture that the liftings of the neighbor syzygies generate the syzygy module of a border basis. We close the paper with a partial result to support this conjecture (see Proposition 30).

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2 Border Division

In the following we use the notation and definitions introduced in [3]. In particular, we work in the polynomial ring $P = K[x_1, \dots, x_n]$ over a field K. The monoid of terms (or monomials or power products) of P is denoted by $\mathbb{T}^n$. For every $d \geq 0$, we let $\mathbb{T}^n_d$ be the set of terms of degree d.

Definition 1 A non-empty set of terms $\mathcal{O} \subseteq \mathbb{T}^n$ is called an **order ideal** if $t \in \mathcal{O}$ implies $t' \in \mathcal{O}$ for every term t' dividing t. The **border** of $\mathcal{O}$ is the set of terms

$$\partial\mathcal{O} = \mathbb{T}^n_1 \cdot \mathcal{O} \ \setminus \ \mathcal{O} = (x_1\mathcal{O} \cup \cdots \cup x_n\mathcal{O}) \setminus \mathcal{O}$$

and the **first border closure** of $\mathcal{O}$ is $\overline{\partial\mathcal{O}} = \mathcal{O} \cup \partial\mathcal{O}$. For every $k \geq 1$, we inductively define the $(k+1)^{\mathrm{st}}$ border $\partial^{k+1}\mathcal{O} = \partial(\overline{\partial^k\mathcal{O}})$ and the $(k+1)^{\mathrm{st}}$ border closure $\overline{\partial^{k+1}\mathcal{O}} = \overline{\partial^k\mathcal{O}} \cup \partial^{k+1}\mathcal{O}$. Finally, we let $\partial^0\mathcal{O} = \overline{\partial^0\mathcal{O}} = \mathcal{O}$.

The following proposition is shown in [3, Proposition 3.4]. It contains three ubiquitous consequences of this definition.

Proposition 2 *Let $\mathcal{O}$ be an order ideal.*

a) For every $k \geq 1$, we have $\partial^k\mathcal{O} = \mathbb{T}^n_k \cdot \mathcal{O} \setminus \mathbb{T}^n_{<k} \cdot \mathcal{O}$.

b) For every $k \geq 1$, we have a disjoint union $\overline{\partial^k\mathcal{O}} = \bigcup_{i=0}^k \partial^i\mathcal{O}$. In particular, we have a disjoint union $\mathbb{T}^n = \bigcup_{i=0}^\infty \partial^i\mathcal{O}$.

c) A term $t \in \mathbb{T}^n$ is divisible by a term in $\partial\mathcal{O}$ if and only if $t \in \mathbb{T}^n \setminus \mathcal{O}$.

In view of this result, we define $\mathrm{ind}_{\mathcal{O}}(t) = \min\{k \geq 0 \mid t \in \overline{\partial^k\mathcal{O}}\}$ for every term $t \in \mathbb{T}^n$ and call it the **index** of t with respect to $\mathcal{O}$. Given a non-zero polynomial $f = c_1t_1 + \cdots + c_st_s \in P$, where $c_1, \dots, c_s \in K \setminus \{0\}$ and $t_1, \dots, t_s \in \mathbb{T}^n$, we order the terms in the support of f such that $\mathrm{ind}_{\mathcal{O}}(t_1) \geq \mathrm{ind}_{\mathcal{O}}(t_2) \geq \cdots \geq \mathrm{ind}_{\mathcal{O}}(t_s)$. Then we call $\mathrm{ind}_{\mathcal{O}}(f) = \mathrm{ind}_{\mathcal{O}}(t_1)$ the **index** of f. The following basic properties of the index were shown in [3, Proposition 3.6]. Note how the two concepts index and degree are complementing one another.

Proposition 3 *Let $\mathcal{O}$ be an order ideal, let $t, t' \in \mathbb{T}^n$, and let $f, g \in P \setminus \{0\}$.*

a) The index $\mathrm{ind}_{\mathcal{O}}(t)$ is the smallest natural number k such that $t = t_1t_2$ with $t_1 \in \mathcal{O}$ and $t_2 \in \mathbb{T}^n_k$.

b) $\mathrm{ind}_{\mathcal{O}}(t\,t') \leq \deg(t) + \mathrm{ind}_{\mathcal{O}}(t')$

c) If $f + g \neq 0$, then $\mathrm{ind}_{\mathcal{O}}(f+g) \leq \max\{\mathrm{ind}_{\mathcal{O}}(f), \mathrm{ind}_{\mathcal{O}}(g)\}$.

d) $\mathrm{ind}_{\mathcal{O}}(fg) \leq \max\{\deg(f) + \mathrm{ind}_{\mathcal{O}}(g),\ \deg(g) + \mathrm{ind}_{\mathcal{O}}(f)\}$

The index and the border form possess properties resembling those of term orderings and leading terms. However, index inequalities need not be preserved under multiplication.

Example 4 Let $P = \mathbb{Q}[x,y]$ and $\mathcal{O} = \{1, x, x^2, x^3, y, y^2\}$. Clearly, the set $\mathcal{O}$ is an order ideal. Its border is $\partial\mathcal{O} = \{x^4, x^3y, x^2y, xy, xy^2, y^3\}$. Hence we have $\mathrm{ind}_{\mathcal{O}}(y^2) = 0 < 1 = \mathrm{ind}_{\mathcal{O}}(xy)$, but, multiplying both sides by x, we get $\mathrm{ind}_{\mathcal{O}}(xy^2) = 1 = \mathrm{ind}_{O}(x^2y)$.

This example also shows that the decomposition $P = \bigoplus_{i\geq 0}(\bigoplus_{\{t|\mathrm{ind}_{\mathcal{O}}(t)=i\}} K \cdot t)$ does not endow P with the structure of a graded ring. Nevertheless the index provides a distance of a term from the order ideal as well as a partial ordering of terms. It allows us to substitute the usual Division Algorithm using a term ordering (see, for instance, [4, Theorem 1.6.4]) by a border version using a partial ordering. For this purpose we introduce the following preliminary notion of a border basis.

Definition 5 Given an order ideal $\mathcal{O} \subseteq \mathbb{T}^n$ with border $\partial\mathcal{O} = \{b_1, \ldots, b_\nu\}$, a set of polynomials $\{g_1, \ldots, g_\nu\} \subseteq P$ is called an **$\mathcal{O}$-border prebasis** if the polynomials have the form $g_i = b_i + h_i$ such that $h_i \in P$ satisfies $\mathrm{Supp}(h_i) \subseteq \mathcal{O}$ for $i = 1, \ldots, \nu$.

Proposition 6 (The Border Division Algorithm)
Let $\mathcal{O} = \{t_1, \ldots, t_\mu\}$ be an order ideal, let $\partial\mathcal{O} = \{b_1, \ldots, b_\nu\}$ be its border, and let $\{g_1, \ldots, g_\nu\}$ be an $\mathcal{O}$-border prebasis. Given a polynomial $f \in P$, consider the following instructions.

D1. Let $f_1 = \cdots = f_\nu = 0$, $c_1 = \cdots = c_\mu = 0$, and $h = f$.

D2. If $h = 0$, then return $(f_1, \ldots, f_\nu, c_1, \ldots, c_\mu)$ and stop.

D3. If $\mathrm{ind}_{\mathcal{O}}(h) = 0$, then find $c_1, \ldots, c_\mu \in K$ such that $h = c_1t_1 + \cdots + c_\mu t_\mu$. Return $(f_1, \ldots, f_\nu, c_1, \ldots, c_\mu)$ and stop.

D4. If $\mathrm{ind}_{\mathcal{O}}(h) > 0$, then let $h = a_1h_1 + \cdots + a_sh_s$ with $a_1, \ldots, a_s \in K\setminus\{0\}$ and $h_1, \ldots, h_s \in \mathbb{T}^n$ such that $\mathrm{ind}_{\mathcal{O}}(h_1) = \mathrm{ind}_{\mathcal{O}}(h)$. Determine the smallest index $i \in \{1, \ldots, \nu\}$ such that h_1 factors as $h_1 = t'\,b_i$ with a term t' of degree $\mathrm{ind}_{\mathcal{O}}(h) - 1$. Subtract $a_1t'g_i$ from h, add a_1t' to f_i, and continue with step D2.

This is an algorithm that returns a tuple $(f_1, \ldots, f_\nu, c_1, \ldots, c_\mu) \in P^\nu \times K^\mu$ such that

$$f = f_1g_1 + \cdots + f_\nu g_\nu + c_1t_1 + \cdots + c_\mu t_\mu$$

and $\deg(f_i) \leq \mathrm{ind}_{\mathcal{O}}(f) - 1$ for all $i \in \{1, \ldots, \nu\}$ with $f_ig_i \neq 0$. This representation does not depend on the choice of the term h_1 in Step D4.

Proof. First we show that the instructions can be executed. In Step D3 the fact that $\mathrm{ind}_{\mathcal{O}}(h) = 0$ implies that all terms in the support of h have index zero, i.e. that they are all in $\mathcal{O}$. In Step D4 we write h as a K-linear combination of terms and note that at least one of them, say h_1, has to have index $k = \mathrm{ind}_{\mathcal{O}}(h)$. By Proposition 3.a, there is a factorization $h_1 = \tilde{t}\,t_i$ for some term $\tilde{t}$

of degree k and some $t_i \in \mathcal{O}$, and there is no such factorization with a term $\tilde{t}$ of smaller degree. Since $k > 0$, we can write $\tilde{t} = t'\, x_j$ for some $t' \in \mathbb{T}^n$ and $j \in \{1, \ldots, n\}$. Then we have $\deg(t') = k - 1$, and the fact that $\tilde{t}$ has the smallest possible degree implies $x_j\, t_i \in \partial\mathcal{O}$. Thus we have $h_1 = t'\,(x_j\, t_i) = t'\, b_k$ for some $b_k \in \partial\mathcal{O}$.

Next we prove termination. We show that Step D4 is performed only finitely many times. Let us investigate the subtraction $h - a_1 t' g_i$ in Step D4. Using Definition 5, we find a representation $g_i = b_i - \sum_{k=1}^{\mu} \alpha_{ki} t_k$ such that $\alpha_{ki} \in K$ for $k = 1, \ldots, \mu$. Hence the subtraction becomes

$$h - a_1 t' g_i = a_1 h_1 + \ldots + a_s h_s - a_1 t' b_i + a_1 t' \sum_{k=1}^{\mu} \alpha_{ki} t_k.$$

Now $a_1 h_1 = a_1 t' b_i$ shows that a term of index $\mathrm{ind}_{\mathcal{O}}(h)$ is removed from h and replaced by terms of the form $t'\, t_\ell \in \overline{\partial^{k-1}\mathcal{O}}$ which have strictly smaller index. The algorithm terminates after finitely many steps because, for a given term, there are only finitely many terms of smaller or equal index.

Finally, we prove correctness. To do so, we show that the equation

$$f = \; h \; + f_1 g_1 + \cdots + f_\nu g_\nu + c_1 t_1 + \cdots + c_\mu t_\mu$$

is an invariant of the algorithm. It is satisfied at the end of Step D1. A polynomial f_i is only changed in Step D4. There the subtraction $h - a_1 t' g_i$ is compensated by the addition $(f_i + a_1 t') g_i$. The constants $c_1, \ldots, c_\mu$ are only changed in Step D3 in which h is replaced by the expression $c_1 t_1 + \ldots + c_\mu t_\mu$. When the algorithm stops, we have $h = 0$. This proves the stated representation of f.

The additional claim that this representation does not depend on the choice of h_1 in Step D4 follows from the observation that h_1 is replaced by terms of strictly smaller index. Thus the different executions of Step D4 corresponding to the reduction of several terms of a given index in h do not interfere with one another, and the final result – after all those terms have been rewritten – is independent of the order in which they are taken care of. □

Notice that in Step D4 the algorithm uses a representation of h that is not necessarily unique due to the partial aspect of the ordering. Also there may be several factorizations $h_1 = \tilde{t}\, t_i$. We choose the index i minimally to determine this step of the algorithm uniquely, but this particular choice is not forced upon us. Finally, the result of the division depends on the numbering of the elements of $\partial\mathcal{O}$, as the next example shows.

Example 7 Let $P = \mathbb{Q}[x, y]$ and let $\mathcal{O} = \{1, x\}$. The border of $\mathcal{O}$ is $\partial\mathcal{O} = \{b_1, b_2, b_3\}$ with $b_1 = y$, $b_2 = xy$, and $b_3 = x^2$. We apply the Border

Division Algorithm to divide $f = x^2y + x^2 + 2xy$ by (g_1, g_2, g_3), where $g_1 = y$, $g_2 = xy - 1$, and $g_3 = x^2 - 1$. The step by step computations are:

D1. Let $f_1 = f_2 = f_3 = 0$ and $c_1 = c_2 = 0$ as well as $h = f$.

D4. Since $\mathrm{ind}_{\mathcal{O}}(h) = 2$, we have $h_1 = x^2y = x^2\, b_1$. Thus we put $h = x^2 + 2xy$ and $f_1 = xy$.

D4. Since $\mathrm{ind}_{\mathcal{O}}(h) = 1$, we choose $h_1 = x^2 = b_3$ and put $h = 2xy + 1$ as well as $f_3 = 1$.

D4. Since $\mathrm{ind}_{\mathcal{O}}(h) = 1$, we have $h_1 = xy = b_2$ and put $h = 3$ as well as $f_2 = 2$.

D3. The algorithm returns $(xy, 2, 1, 3, 0)$.

Therefore, there is a representation

$$f = x^2\,(y) + 2\,(xy - 1) + (x^2 - 1) + 3 = x^2\, g_1 + 2\, g_2 + g_3 + 3.$$

However, when we apply the algorithm to the shuffled tuple (g_1', g_2', g_3) where $g_1' = g_2$ and $g_2' = g_1$, it computes the representation

$$f = (x + 2)\,(xy - 1) + (x^2 - 1) + 3 + x = (x + 2)\, g_1' + 0 \cdot g_2' + g_3 + 3 + x.$$

If we fix the tuple $\mathcal{G} = (g_1, \ldots, g_\nu)$, the result of the Border Division Algorithm is uniquely determined. This we do. Given an order ideal $\mathcal{O} = \{t_1, \ldots, t_\mu\}$ and a polynomial $f \in P$, let $f = f_1g_1 + \cdots + f_\nu g_\nu + c_1t_1 + \cdots + c_\mu t_\mu$ be a representation computed by the Border Division Algorithm. Then $\mathrm{NR}_{\mathcal{O},\mathcal{G}}(f) = c_1t_1 + \cdots + c_st_s$ is called the **normal $\mathcal{O}$-remainder** of f. As we saw in the example, the normal $\mathcal{O}$-remainder sometimes depends on the order of the elements in $\mathcal{G}$.

By construction, a polynomial f and $\mathrm{NR}_{\mathcal{O},\mathcal{G}}(f)$ represent the same residue class modulo $(g_1, \ldots, g_\nu)$. This shows that the residue classes of the terms in $\mathcal{O}$ generate $P/(g_1, \ldots, g_\nu)$ as a K-vector space. However, they do not necessarily constitute a K-basis of this vector space.

In particular, if $\mathcal{O}$ consists of finitely many terms, then the ideal $(g_1, \ldots, g_\nu)$ generated by an $\mathcal{O}$-border prebasis is zero-dimensional.

3 Characterizations of Border Bases

First we prescribe the setting that is used throughout the remainder of this paper. Let $\mathcal{O} = \{t_1, \ldots, t_\mu\}$ be an order ideal consisting of finitely many terms. We let $\partial\mathcal{O} = \{b_1, \ldots, b_\nu\}$ be its border, $G = \{g_1, \ldots, g_\nu\}$ an $\mathcal{O}$-border prebasis, and I a zero-dimensional ideal of P containing G. In this setting, a border basis is defined as follows.

Definition 8 The $\mathcal{O}$-border prebasis $\{g_1, \dots, g_\nu\}$ is called an *$\mathcal{O}$-border basis* of I if the residue classes of the elements of $\mathcal{O}$ form a K-vector space basis of P/I.

If I has an $\mathcal{O}$-border basis G, then it is uniquely determined by $\mathcal{O}$ and G generates I [3, Section 4.1]. According to the remarks at the end of the previous section a border prebasis is a border basis if and only if the residue classes of $t_1, \dots, t_\mu$ modulo I are K-linearly independent or, equivalently, $I \cap \langle \mathcal{O} \rangle_K = 0$.

We are going to develop the theory of border bases in analogy with the development of the theory of Gröbner bases in [4, Chapter 2]. Hence, we shall prove characterizations of border bases which imitate the characterizations of Gröbner bases given there. Our first result is a border basis version of the so-called **special generation** property.

Proposition 9 (Border Bases and Special Generation)
In the prescribed setting, the set G is an $\mathcal{O}$-border basis of I if and only if one of the following equivalent conditions is satisfied.

A_1. *For every $f \in I \setminus \{0\}$, there exist polynomials $f_1, \dots, f_\nu \in P$ such that $f = f_1 g_1 + \dots + f_\nu g_\nu$ and $\deg(f_i) \leq \mathrm{ind}_{\mathcal{O}}(f) - 1$ whenever $f_i g_i \neq 0$.*

A_2. *For every $f \in I \setminus \{0\}$, there exist polynomials $f_1, \dots, f_\nu \in P$ such that $f = f_1 g_1 + \dots + f_\nu g_\nu$ and $\max\{\deg(f_i) \mid i \in \{1, \dots, \nu\},\ f_i g_i \neq 0\} = \mathrm{ind}_{\mathcal{O}}(f) - 1$.*

Proof. First we show that A_1 holds if G is an $\mathcal{O}$-border basis. The Border Division Algorithm computes a representation $f = f_1 g_1 + \dots + f_\nu g_\nu + c_1 t_1 + \dots + c_\mu t_\mu$ with $f_1, \dots, f_\nu \in P$ and $c_1, \dots, c_\mu \in K$ such that $\deg(f_i) \leq \mathrm{ind}_{\mathcal{O}}(f) - 1$ for $i = 1, \dots, \nu$. Then $c_1 t_1 + \dots + c_\mu t_\mu \equiv 0 \pmod{I}$, and the hypothesis implies $c_1 = \dots = c_\mu = 0$.

Next we prove that A_1 implies A_2. If $\deg(f_i) < \mathrm{ind}_{\mathcal{O}}(f) - 1$, then Proposition 3.b yields $\mathrm{ind}_{\mathcal{O}}(f_i g_i) \leq \deg(f_i) + \mathrm{ind}_{\mathcal{O}}(g_i) = \deg(f_i) + 1 < \mathrm{ind}_{\mathcal{O}}(f)$. By Proposition 3.c, there has to be at least one number $i \in \{1, \dots, \nu\}$ such that $\deg(f_i) = \mathrm{ind}_{\mathcal{O}}(f) - 1$.

Finally, assume A_2 and let $c_1, \dots, c_\mu \in K$ satisfy $c_1 t_1 + \dots + c_\mu t_\mu \in I$. Then either $f = c_1 t_1 + \dots + c_\mu t_\mu$ equals the zero polynomial or not. In the latter case we apply the first part of A_2 to obtain a representation $f = f_1 g_1 + \dots + f_\nu g_\nu$ with $f_1, \dots, f_\nu \in P$. Since $f \neq 0$, we have $\max\{\deg(f_i) \mid i \in \{1, \dots, \nu\},\ f_i g_i \neq 0\} \geq 0$. But $\mathrm{ind}_{\mathcal{O}}(f) - 1 = -1$, which contradicts the second part of A_2. Hence, f is zero, $I \cap \langle \mathcal{O} \rangle_K = 0$, and the set G is an $\mathcal{O}$-border basis. □

Commonly, Gröbner bases are defined as sets of polynomials whose leading terms generate the leading term ideal. In the theory of border bases, leading terms have to be replaced with more general border forms which are defined as follows.

Definition 10 Given a polynomial $f \in P$, there is a representation $f = a_1 u_1 + \cdots + a_s u_s$ with $a_1, \ldots, a_s \in K \setminus \{0\}$ and $u_1, \ldots, u_s \in \mathbb{T}^n$ such that $\operatorname{ind}_{\mathcal{O}}(u_1) \geq \cdots \geq \operatorname{ind}_{\mathcal{O}}(u_s)$.

a) The polynomial
$$\mathrm{BF}_{\mathcal{O}}(f) = \sum_{\{i \mid \operatorname{ind}(u_i) = \operatorname{ind}(f)\}} a_i u_i$$
is called the **border form** of f with respect to $\mathcal{O}$. For $f = 0$, we let $\mathrm{BF}_{\mathcal{O}}(f) = 0$.

b) Given an ideal $I \subseteq P$, the ideal $\mathrm{BF}_{\mathcal{O}}(I) = (\mathrm{BF}_{\mathcal{O}}(f) \mid f \in I)$ is called the **border form ideal** of I with respect to $\mathcal{O}$.

The definition is independent of the chosen representation. As an important example, the elements of the $\mathcal{O}$-border prebasis G have the border form $\mathrm{BF}_{\mathcal{O}}(g_i) = b_i$; in particular, they consist of only one term. Now we characterize border bases by their border form ideal.

Proposition 11 (Border Bases and the Border Form Ideal) *In the prescribed setting, the set G is an $\mathcal{O}$-border basis of I if and only if one of the following equivalent conditions is satisfied.*

B_1. *For every $f \in I$, the support of $\mathrm{BF}_{\mathcal{O}}(f)$ is contained in $\mathbb{T}^n \setminus \mathcal{O}$.*

B_2. *We have $\mathrm{BF}_{\mathcal{O}}(I) = (b_1, \ldots, b_\nu)$.*

Proof. First we show that a border basis satisfies condition B_1. Suppose that the border form of a polynomial $f \in I \setminus \{0\}$ contains a term of $\mathcal{O}$ in its support. Then all terms in the support of f are contained in $\mathcal{O}$, i.e. $f = c_1 t_1 + \cdots + c_\mu t_\mu$ for suitable $c_1, \ldots, c_\mu \in K$. The border basis hypothesis implies $c_1 = \cdots = c_\mu = 0$, which contradicts $f \neq 0$.

Next we prove that B_1 implies B_2. Since $g_i \in I$, we have $b_i = \mathrm{BF}_{\mathcal{O}}(g_i) \in \mathrm{BF}_{\mathcal{O}}(I)$ for $i = 1, \ldots, \nu$. To prove the reverse inclusion, let $f \in I \setminus \{0\}$. By B_1 and Proposition 2.c, every term in the support of $\mathrm{BF}_{\mathcal{O}}(f)$ is divisible by a term in $\partial\mathcal{O}$. Hence the border form of f is contained in $(b_1, \ldots, b_\nu)$.

Finally, we show that B_2 implies that G is a border basis. Let $c_1, \ldots, c_\mu \in K$ be elements such that $f = c_1 t_1 + \cdots + c_\mu t_\mu \in I$. Then all terms in the support of f have index zero, and thus $f = \mathrm{BF}_{\mathcal{O}}(f)$. So, B_2 and Proposition 2.c imply $c_1 = \cdots = c_\mu = 0$. □

To characterize border bases in analogy with conditions $C_1) - C_4)$ of [4, Section 2.2] we define the rewrite relation associated to G. Let $f \in P$ be a polynomial such that $t \in \operatorname{Supp}(f)$ is a multiple of a border term $t = t' b_i$.

Let $c \in K$ be the coefficient of t in f. Then $h = f - ct'g_i$ does not contain the term t anymore. We say that f **reduces to** h **in one step** using g_i and write $f \xrightarrow{g_i} h$. (Instead one may consider the more restrictive rewrite rule that, in addition, the factorization $t = t'b_i$ must be optimal in the sense $\operatorname{ind}(t) = \deg(t') + 1$. For instance, the reduction steps used in the Border Division Algorithm satisfy this additional condition. However, the results below indicate that our less restrictive rewrite rule is an appropriate choice.) The reflexive, transitive closure of the relations $\xrightarrow{g_i}$, $i \in \{1, \ldots, \nu\}$, is called the **rewrite relation** associated to G and is denoted by $\xrightarrow{G}$. The equivalence relation generated by $\xrightarrow{G}$ is denoted by $\xleftrightarrow{G}$. In stark contrast to Gröbner basis theory, rewrite relations associated to border prebasis are, in general, not Noetherian; this is demonstrated by the following example.

Example 12 Let $P = \mathbb{Q}[x, y]$ and $\mathcal{O} = \{1, x, y, x^2, y^2\}$. Then $\mathcal{O}$ is an order ideal with border $\partial\mathcal{O} = \{xy, x^3, x^2y, xy^2, y^3\}$. Consider the $\mathcal{O}$-border prebasis $G = \{g_1, \ldots, g_5\}$, where $g_1 = xy - x^2 - y^2$, $g_2 = x^3$, $g_3 = x^2y$, $g_4 = xy^2$, and $g_5 = y^3$. The chain of reductions

$$x^2y \xrightarrow{g_1} x^3 + xy^2 \xrightarrow{g_2} xy^2 \xrightarrow{g_1} x^2y + y^3 \xrightarrow{g_5} x^2y$$

can be repeated indefinitely, and hence $\xrightarrow{G}$ is not Noetherian.

The following properties of the equivalence relation $\xleftrightarrow{G}$ can be proved in exactly the same way as the corresponding properties in Gröbner basis theory (cf. [4, Proposition 2.2.2]).

Proposition 13 *Let $\xleftrightarrow{G}$ be the rewrite equivalence relation associated to an $\mathcal{O}$-border prebasis $G = \{g_1, \ldots, g_\nu\}$, and let $f_1, f_2, f_3, f_4 \in P$.*

a) If $f_1 \xleftrightarrow{G} f_2$ and $f_3 \xleftrightarrow{G} f_4$, then $f_1 + f_3 \xleftrightarrow{G} f_2 + f_4$.

b) If $f_1 \xleftrightarrow{G} f_2$, then $f_1f_3 \xleftrightarrow{G} f_2f_3$.

c) We have $f_1 \xleftrightarrow{G} f_2$ if and only if $f_1 - f_2 \in (g_1, \ldots, g_\nu)$.

According to property c) the rewrite equivalence relation $\xleftrightarrow{G}$ is in fact the congruence relation modulo the ideal $(g_1, \ldots, g_\nu)$. In other words, applying reduction steps forwards as well as backwards to a polynomial, we can move through the complete congruence class modulo $(g_1, \ldots, g_\nu)$ in search for a "good" representative.

Regardless of their lack of Noetherianity, rewrite relations $\xrightarrow{G}$ characterize border bases by the confluence property. In this respect $\xrightarrow{G}$ is called **confluent** if for any two co-initial reductions $f_1 \xrightarrow{G} f_2$ and $f_1 \xrightarrow{G} f_3$ there exist co-terminal reductions $f_2 \xrightarrow{G} f_4$ and $f_3 \xrightarrow{G} f_4$. Finally, a polynomial $f \in P$

is called G**-reduced** if $f \xrightarrow{G} h$ implies $h = f$.

For example, any polynomial f with support in $\mathcal{O}$ is G-reduced; by Proposition 2.c, it cannot contain a term that can be reduced. In particular, the normal remainder $\mathrm{NR}_{\mathcal{O},\mathcal{G}}(f)$ computed by the Border Division Algorithm is G-reduced.

Proposition 14 (Border Bases and Rewrite Relations)
In the prescribed setting, the set G is an $\mathcal{O}$-border basis of I if and only if one of the following equivalent conditions is satisfied.

C_1. *For $f \in P$, we have $f \xrightarrow{G} 0$ if and only if $f \in I$.*

C_2. *If $f \in I$ is G-reduced, then $f = 0$.*

C_3. *For every $f \in P$, there exists a G-reduced element $h \in P$ such that $f \xrightarrow{G} h$ and h is unique.*

C_4. *The rewrite relation $\xrightarrow{G}$ is confluent.*

Proof. First we show that a border basis has property C_1. If a polynomial $f \in P$ satisfies $f \xrightarrow{G} 0$, then it is enough to collect the subtractions performed by the individual reduction steps on the right-hand side to obtain $f \in (g_1, \dots, g_\nu)$. Conversely, let $f \in I$. We apply the Border Division Algorithm to f; it performs reduction steps using elements of G to compute the normal remainder $\mathrm{NR}_{\mathcal{O},\mathcal{G}}(f) \in \langle \mathcal{O} \rangle_K$. Since $f \in I$, we also have $\mathrm{NR}_{\mathcal{O},\mathcal{G}}(f) \in I$. The hypothesis that G is a border basis yields $\mathrm{NR}_{\mathcal{O},\mathcal{G}}(f) \in I \cap \langle \mathcal{O} \rangle_K = 0$, i.e. $f \xrightarrow{G} 0$.

To prove that C_1 implies C_2, note that C_1 shows $f \xrightarrow{G} 0$ for $f \in I$. Thus a G-reduced polynomial $f \in I$ has to be zero. Next we prove that C_2 implies C_3. Let $f \in P$. The Border Division Algorithm performs a reduction $f \xrightarrow{G} \mathrm{NR}_{\mathcal{O},\mathcal{G}}(f)$, i.e. there exists a reduction to a G-reduced polynomial. Suppose that $f \xrightarrow{G} h$ and h is G-reduced. Then $h - \mathrm{NR}_{\mathcal{O},\mathcal{G}}(f) \in I$ and the support of this difference is contained in $\mathcal{O}$. Thus it is G-reduced and C_2 yields $h = \mathrm{NR}_{\mathcal{O},\mathcal{G}}(f)$. Altogether, the normal remainder of f has the properties required by C_3.

Now we show that C_3 implies C_4. Let $f_1 \xrightarrow{G} f_2$ and $f_1 \xrightarrow{G} f_3$ be co-initial reductions. The Border Division Algorithm produces $f_1 \xrightarrow{G} \mathrm{NR}_{\mathcal{O},\mathcal{G}}(f_2)$ and $f_1 \xrightarrow{G} \mathrm{NR}_{\mathcal{O},\mathcal{G}}(f_3)$. Since normal remainders are G-reduced, condition C_3 implies $\mathrm{NR}_{\mathcal{O},\mathcal{G}}(f_2) = \mathrm{NR}_{\mathcal{O},\mathcal{G}}(f_3)$. Therefore, there are co-terminal reductions $f_2 \xrightarrow{G} f_4$ and $f_3 \xrightarrow{G} f_4$ with $f_4 = \mathrm{NR}_{\mathcal{O},\mathcal{G}}(f_2) = \mathrm{NR}_{\mathcal{O},\mathcal{G}}(f_3)$.

Finally, to show that G is a border basis if it satisfies C_4, we can use Proposition 13.c and proceed as in the proof of $C_4) \Rightarrow C_1)$ in [4, Proposition 2.2.5]. $\square$

Given an $\mathcal{O}$-border basis $G = \{g_1, \ldots, g_\nu\}$ and $f \in P$, the unique G-reduced polynomial h such that $f \xrightarrow{G} h$ is the normal remainder $\mathrm{NR}_{\mathcal{O},\mathcal{G}}(f)$. Hence it is effectively computed by the Border Division Algorithm and it agrees with the **normal form** $\mathrm{NF}_{\mathcal{O},I}(f)$ with respect to the ideal $I = (g_1, \ldots, g_\nu)$. The properties of the normal form are studied in [3, Section 4.2].

4 A Buchberger Criterion for Border Bases

Instead of examining a polynomial ideal I directly, one can consider its quotient algebra P/I. The K-vector space structure of P/I suffices to single out the zero-dimensional ideals, as they are precisely those ideals with a finite-dimensional quotient. Now each multiplication by an element of P/I defines a K-linear map and thus we obtain a P-module structure on P/I. This P-module structure determines the ideal I as its annihilator. In particular, the indeterminates $x_1, \ldots x_n$ define so-called mutliplication matrices which commute. This procdure can be reversed in the following sense. For each border prebasis we define formal multiplication matrices. Then the border prebasis is a border basis if and only if the formal multiplication matrices commute. A detailed account of these remarks is given in [3].

Definition 15 Let $\mathcal{O} = \{t_1, \ldots, t_\mu\}$ be an order ideal, $\partial\mathcal{O} = \{b_1, \ldots, b_\nu\}$ its border, and $G = \{g_1, \ldots, g_\nu\}$ an $\mathcal{O}$-border prebasis with

$$g_j = b_j - \sum_{m=1}^{\mu} \alpha_{mj} t_m \qquad , \quad 1 \leq j \leq \nu$$

For $1 \leq r \leq n$, define the r**-th formal multiplication matrix** $\mathcal{X}_r := (\xi_{k\ell}^{(r)})$ by

$$\xi_{k\ell}^{(r)} = \begin{cases} \delta_{ki} & , \text{if } t_i = x_r\, t_\ell \\ \alpha_{kj} & , \text{if } b_j = x_r\, t_\ell \end{cases}.$$

The formal multiplication matrices encode the following procedure. First, we multiply an element of $\langle \mathcal{O} \rangle_K$ by the indeterminate x_r and, second, whenever $x_r\, t_i = b_j$ is a border term, we reduce by the corresponding border polynomial g_j. The reduction guarantees that the result stays in $\langle \mathcal{O} \rangle_K$. More concretely, the elements $c_1\, t_1 + \ldots + c_\mu\, t_\mu \in \langle \mathcal{O} \rangle_K$ are encoded as column vectors $c_1\, e_1 + \ldots + c_\mu\, e_\mu \in K^\mu$. For example, $x_r\, t_i$ corresponds to $\mathcal{X}_r\, (c_1, \ldots, c_\mu)^T$.

Bernard Mourrain [6] showed the following result. A proof using our notation and terminology is contained in [3].

Proposition 16 (Border Bases and Formal Multiplication Matrices) *In the setting of Definition 15, the border prebasis G is a border basis if and only if the formal multiplication matrices commute, i.e. if and only if $\mathcal{X}_r\,\mathcal{X}_s = \mathcal{X}_s\,\mathcal{X}_r$ for all $r, s \in \{1, \ldots, n\}$.*

Next we want to analyze these commutativity conditions in more detail by considering their effect on an arbitrary base vector e_i. For each $i \in \{1, \ldots, \mu\}$ we compare $\mathcal{X}_r\,\mathcal{X}_s\,e_i$ with $\mathcal{X}_s\,\mathcal{X}_r\,e_i$. Translating the comparision back into the language of $\langle \mathcal{O} \rangle_K$, we shall find that the resulting description depends on the position of t_i relative to the border. The following case by case discussion reveals the details.

To lighten the index load we abbreviate $x = x_r$ and $y = x_s$.

$$\begin{array}{|c|c|} \hline t_k & t_l \\ \hline t_i & t_j \\ \hline \end{array}$$

First case: Let $x\,y\,t_i \in \mathcal{O}$.

Since $\mathcal{O}$ is an order ideal, we also have $x\,t_i,\ y\,t_i \in \mathcal{O}$, say $x\,t_i = t_j$, $y\,t_i = t_k$, and $x\,y\,t_i = t_\ell$. Then $\mathcal{X}\,\mathcal{Y}\,e_i = \mathcal{X}\,e_k = e_\ell = \mathcal{Y}\,e_j = \mathcal{Y}\,\mathcal{X}\,t_i$, i.e., the commutativity condition holds by definition of the formal multiplication matrices.

$$\begin{array}{|c|c|} \hline t_k & b_l \\ \hline t_i & t_j \\ \hline \end{array}$$

Second case: Let $x\,y\,t_i \in \partial\mathcal{O}$ and $x\,t_i,\ y\,t_i \in \mathcal{O}$.

Say $x\,t_i = t_j$, $y\,t_i = t_k$, and $x\,y\,t_i = b_\ell$. Then

$$\mathcal{X}\,\mathcal{Y}\,e_i = \mathcal{X}\,e_k = \begin{pmatrix} \alpha_{1\ell} \\ \vdots \\ \alpha_{\mu\ell} \end{pmatrix} = \mathcal{Y}\,e_j = \mathcal{Y}\,\mathcal{X}\,e_i$$

Again, commutativity follows immediately from the definition of the formal multiplication matrices.

$$\begin{array}{|c|c|} \hline b_k & b_l \\ \hline t_i & t_j \\ \hline \end{array}$$

Third case: Let $x\,t_i \in \mathcal{O}$ and $y\,t_i \in \partial\mathcal{O}$.

Since $\overline{\partial\mathcal{O}}$ and $\mathcal{O}$ are order ideals, this case implies $x\,y\,t_i \in \partial\mathcal{O}$. Say $x\,t_i = t_j$, $y\,t_i = b_k$, and $x\,y\,t_i = b_\ell$. The commutativity condition becomes

$$\mathcal{X} \begin{pmatrix} \alpha_{1k} \\ \vdots \\ \alpha_{\mu k} \end{pmatrix} = \begin{pmatrix} \alpha_{1\ell} \\ \vdots \\ \alpha_{\mu\ell} \end{pmatrix}$$

i.e., $\sum_{m=1}^{\mu} \xi_{pm}\alpha_{mk} = \alpha_{p\ell}$ for all $p \in \{1, \ldots, \mu\}$. According to the definition of the formal multiplication matrices, this condition can be rewritten as follows.

$$\sum_{\substack{m \\ x\,t_m = t_{\varphi(m)}}} \delta_{p,\varphi(m)} \alpha_{mk} + \sum_{\substack{m \\ x\,t_m = b_{\psi(m)}}} \alpha_{p,\psi(m)} \alpha_{mk} = \alpha_{p\ell} \quad , \quad 1 \le p \le \mu \tag{1}$$

The first sum stretches over all indices m for which $x\,t_m \in \mathcal{O}$. For such an index m, let $\varphi(m)$ be the index with $x\,t_i = t_{\varphi(m)}$. The notation in the second sum is chosen analogously.

b_k	$*$
t_i	b_j

Fourth case: Let $x\,t_i,\ y\,t_i \in \partial\mathcal{O}$.

Say $x\,t_i = b_j$ and $y\,t_i = b_k$. The commutativity condition becomes

$$\mathcal{X} \begin{pmatrix} \alpha_{1k} \\ \vdots \\ \alpha_{\mu k} \end{pmatrix} = \mathcal{Y} \begin{pmatrix} \alpha_{1j} \\ \vdots \\ \alpha_{\mu j} \end{pmatrix}$$

i.e., $\sum_{m=1}^{\mu} \xi_{pm} \alpha_{mk} = \sum_{m=1}^{\mu} \eta_{pm} \alpha_{mj}$ for all $p \in \{1, \ldots, \mu\}$. This condition can be rewritten as

$$\begin{aligned} &\sum_{\substack{m \\ x\,t_m = t_{\varphi(m)}}} \delta_{p,\varphi(m)} \alpha_{mk} + \sum_{\substack{m \\ x\,t_m = b_{\psi(m)}}} \alpha_{p,\psi(m)} \alpha_{mk} = \\ &\quad \sum_{\substack{m \\ x\,t_m = t_{\varrho(m)}}} \delta_{p,\varrho(m)} \alpha_{mj} + \sum_{\substack{m \\ x\,t_m = b_{\sigma(m)}}} \alpha_{p,\sigma(m)} \alpha_{mj} \quad , \quad 1 \le p \le \mu \end{aligned} \tag{2}$$

This covers all cases. The two cases with non-trivial commutativity conditions motivate the following definition.

Definition 17 Let $b_i, b_j \in \partial\mathcal{O}$ be two distinct border terms.

a) The border terms b_i and b_j are called **next-door neighbors** if $b_i = x\,b_j$ for some $x \in \{x_1, \ldots, x_n\}$.
b) The border terms b_i and b_j are called **across-the-street neighbors** if $x\,b_i = y\,b_j$ for some $x, y \in \{x_1, \ldots, x_n\}$.
c) The border terms b_i and b_j are called **neighbors** if they are next-door neighbors or across-the-street neighbors.

This definition comprises slightly more than the above case distinction. In a polynomial ring with at least three indeterminates there are order ideals that allow a constellation of border terms $b_j = x\,b_i$ and $b_k = y\,b_i$,

b_k	$*$
b_i	b_j

,

which, correctly, is absent from the above cases. The above case by case discussion considers at most the relations $b_j = x\,b_i$ and $b_k = y\,b_i$, while it disregards $x\,b_k = y\,b_j$. Our definition also acknowledges b_k and b_j as neighbors.

In the remainder of this section we interpret the commutativity conditions in terms of rewrite rules.

Consider the next-door neighbor relation $b_\ell - x\,b_k = 0$. The corresponding combination of border polynomials is

$$\begin{aligned} g_\ell - x\,g_k &= (b_\ell - \sum_{m=1}^{\mu} \alpha_{m\ell} t_m) - x(b_k - \sum_{m=1}^{\mu} \alpha_{mk} t_m) \\ &= -\sum_{m=1}^{\mu} \alpha_{m\ell} t_m + \sum_{m=1}^{\mu} \alpha_{mk}(x\,t_m) \\ &= -\sum_{m=1}^{\mu} \alpha_{m\ell} t_m + \sum_{x\,t_m = t_{\varphi(m)}} \alpha_{mk} t_{\varphi(m)} + \sum_{x\,t_m = b_{\psi(m)}} \alpha_{mk} b_{\psi(m)} \\ &= -\sum_{m=1}^{\mu} \alpha_{m\ell} t_m + \sum_{x\,t_m = t_{\varphi(m)}} \alpha_{mk} t_{\varphi(m)} + \sum_{x\,t_m = b_{\psi(m)}} \alpha_{mk} g_{\psi(m)} \\ &\quad + \sum_{x\,t_m = b_{\psi(m)}} (\alpha_{mk} \sum_{s=1}^{\mu} \alpha_{s,\psi(m)} t_s) \end{aligned}$$

Since $(g_1, \ldots, g_\nu) \subseteq I$, we obtain the congruence

$$0 \equiv -\sum_{m=1}^{\mu} \alpha_{m\ell} t_m + \sum_{x\,t_m = t_{\varphi(m)}} \alpha_{mk} t_{\varphi(m)} + \sum_{x\,t_m = b_{\psi(m)}} (\alpha_{mk} \sum_{s=1}^{\mu} \alpha_{s,\psi(m)} t_s) \pmod{I}.$$

For a border basis the coefficient of each t_p, $1 \le p \le \mu$, on the right-hand side must vanish. This vanishing condition is exactly the commutativity condition (1).

Across-the-street neighbor combinations $x\,g_k - y\,g_j$ are treated analogously. The corresponding combination of border polynomials is

$$\begin{aligned} x\,g_k - y\,g_j &= x(b_k - \sum_{m=1}^{\mu} \alpha_{mk} t_m) - y(b_j - \sum_{m=1}^{\mu} \alpha_{mj} t_m) \\ &= -\sum_{m=1}^{\mu} \alpha_{mk}(x\,t_m) + \sum_{m=1}^{\mu} \alpha_{mj}(y\,t_m) \end{aligned}$$

$$
\begin{aligned}
&= - \sum_{x\,t_m = t_{\varphi(m)}} \alpha_{mk} t_{\varphi(m)} - \sum_{x\,t_m = b_{\psi(m)}} \alpha_{mk} b_{\psi(m)} \\
&\quad + \sum_{y\,t_m = t_{\varrho(m)}} \alpha_{mj} t_{\varrho(m)} + \sum_{y\,t_m = b_{\sigma(m)}} \alpha_{mj} b_{\sigma(m)} \\
&= - \sum_{x\,t_m = t_{\varphi(m)}} \alpha_{mk} t_{\varphi(m)} - \sum_{x\,t_m = b_{\psi(m)}} \alpha_{mk} g_{\psi(m)} \\
&\quad - \sum_{x\,t_m = b_{\psi(m)}} \alpha_{mk} \sum_{s=1}^{\mu} \alpha_{s\psi(m)} t_s \\
&\quad + \sum_{y\,t_m = t_{\varrho(m)}} \alpha_{mj} t_{\varrho(m)} + \sum_{y\,t_m = b_{\sigma(m)}} \alpha_{mj} g_{\sigma(m)} \\
&\quad + \sum_{y\,t_m = b_{\sigma(m)}} \alpha_{mj} \sum_{s=1}^{\mu} \alpha_{s\sigma(m)} t_s
\end{aligned}
$$

We obtain the congruence

$$
\begin{aligned}
0 \equiv &- \sum_{x\,t_m = t_{\varphi(m)}} \alpha_{mk} t_{\varphi(m)} - \sum_{x\,t_m = b_{\psi(m)}} \alpha_{mk} \sum_{s=1}^{\mu} \alpha_{s\psi(m)} t_s \\
&+ \sum_{y\,t_m = t_{\varrho(m)}} \alpha_{mj} t_{\varrho(m)} + \sum_{y\,t_m = b_{\sigma(m)}} \alpha_{mj} \sum_{s=1}^{\mu} \alpha_{s\sigma(m)} t_s \pmod{I}
\end{aligned}
$$

Considering the coefficients individually produces the commutativity condition (2).

This computation allows us to characterize border bases analogously to Buchberger's criterion for Gröbner bases. A similar result appears in [7, Theorem 8.11].

In Gröbner basis theory the S-polynomials are obtained by applying a syzygy of two leading terms to the corresponding polynomials. Analogously, we apply the fundamental syzygy of the border terms b_i and b_j and get the corresponding **S-polynomial**. More concretely, an S-polynomial has the form

$$
\mathrm{S}(g_i, g_j) = (\mathrm{lcm}(b_i, b_j)/b_i)\, g_i - (\mathrm{lcm}(b_i, b_j)/b_j)\, g_j .
$$

Proposition 18 (Buchberger Criterion for Border Bases)
In the prescribed setting, the $\mathcal{O}$-border prebasis G is an $\mathcal{O}$-border basis of I if and only if one of the following equivalent conditions is satisfied.

D_1. *For all $1 \le i < j \le \nu$, the S-polynomial $\mathrm{S}(g_i, g_j)$ reduces to zero via $\xrightarrow{G}$.*

D_2. *For all neighbors b_i and b_j, the S-polynomial $\mathrm{S}(g_i, g_j)$ reduces to zero via $\xrightarrow{G}$.*

Proof. Condition D_1 holds if G is a border basis, since $\mathrm{S}(g_i, g_j) \in I$ and G satisfies Condition C_1. Since D_1 logically implies D_2, it remains to prove that G is a border basis if D_2 holds. Let $g_i - xg_j$ or $xg_i - yg_j$ be the S-polynomial corresponding to a neighbor syzygy. The above calculation shows that $g_\ell - x\, g_k \xrightarrow{G} 0$ or $x\, g_k - y\, g_j \xrightarrow{G} 0$ respectively implies the commutativity of the formal multiplication matrices, and therefore that G is an $\mathcal{O}$-border basis. □

Condition D_2 can be rephrased as follows:

a) For all next-door neighbors $b_\ell = x\, b_k$, there are constant coefficients $c_1 \ldots, c_\nu \in K$ such that $g_\ell - x\, g_k = \sum_{j=1}^{\nu} c_j\, g_j$.
b) For all across-the-street neighbors $x\, b_k - y\, b_j$, there are constant coefficients $d_1 \ldots, d_\nu \in K$ such that $x\, g_k - y\, g_j = \sum_{j=1}^{\nu} d_j\, g_j$.

This is the special generation condition restricted to all neighbor combinations. By the preceding Buchberger criterion and the characterization of border bases via special generation, this implies the special generation of all polynomials in the ideal.

5 Border Syzygies

In this section we study the syzygy module

$$\mathrm{Syz}_P(b_1, \ldots, b_\nu) \;=\; \{(f, \ldots, f_\nu) \in P^\nu \mid f_1 b_1 + \cdots + f_\nu b_\nu = 0\}.$$

Its elements are called **border syzygies**.

As preliminary work, we consider the neighboring structure of the border $\partial\mathcal{O}$. Let $\sim$ denote the equivalence relation generated by the neighbor relation. The following proposition states that $\partial\mathcal{O}$ is connected in the sense that there is only one equivalence class with respect to $\sim$.

Proposition 19 *For any two border terms $b_i, b_j \in \partial\mathcal{O}$, there is a finite sequence $b_{k_0}, b_{k_1}, \ldots, b_{k_s}$ of border terms from $b_i = b_{k_0}$ to $b_j = b_{k_s}$ such that $b_{k_{\ell-1}}, b_{k_\ell}$ are neighbors for $\ell = 1, \ldots, s$.*

Proof. Let $b_i, b_j \in \partial\mathcal{O}$ and $g = \gcd(b_i, b_j)$. Then we have $b_i = x_{i_1}^{\alpha_1} \cdots x_{i_p}^{\alpha_p}\, g$ and $b_j = x_{j_1}^{\beta_1} \cdots x_{j_q}^{\beta_q}\, g$ with $\alpha_k, \beta_\ell \in \mathbb{N}_+$ and $\{i_1, \ldots, i_p\} \cap \{j_1, \ldots, j_q\} = \emptyset$. We use induction on $\alpha_1 + \cdots + \alpha_p + \beta_1 + \ldots + \beta_q$. If $\beta_1 + \ldots + \beta_q = 0$, i.e. if $g = b_j$, then

$$b_i = b_{k_0} = x_{i_1}^{\alpha_1} \cdots x_{i_p}^{\alpha_p}\, g = b_i, \quad x_{i_1}^{\alpha_1} \cdots x_{i_p}^{\alpha_p - 1}\, g, \quad \ldots, \quad x_{i_1}^{\alpha_1} \cdots x_{i_{p-1}}^{\alpha_{p-1}}\, g,$$
$$x_{i_1}^{\alpha_1} \cdots x_{i_{p-1}}^{\alpha_{p-1}-1}\, g, \quad \ldots, \quad x_{i_1}\, g, \quad g = b_{k_s} = b_j$$

is a sequence of border terms, since $\overline{\partial \mathcal{O}}$ and $\mathcal{O}$ are order ideals. By construction, any two consecutive terms in this sequence are next-door neighbors. Symmetrically, the case $\alpha_1 + \ldots + \alpha_p = 0$ is proved.

Now assume that $\alpha_1 + \ldots + \alpha_p,\ \beta_1 + \ldots + \beta_q > 0$. Then $x_{j_1} \mid b_j$, say $b_j = x_{j_1} t$ with $t \in \mathbb{T}^n$. Since $\overline{\partial \mathcal{O}}$ is an order ideal, we have $t \in \overline{\partial \mathcal{O}}$. We finish the proof by considering three cases.

Case 1: If $t \in \partial \mathcal{O}$, then t is a next-door neighbor of b_j and $t \sim b_i$ by induction hypothesis. Thus we have $b_i \sim b_j$ in this case.

Case 2: If $t \in \mathcal{O}$ and $x_{i_1} t \in \partial \mathcal{O}$, then $x_{i_1} t$ is an across-the-street neighbor of b_j. Since $\gcd(x_{i_1} t,\, b_i) = x_i\, g$, we have $x_{i_1} t \sim b_i$ by induction hypothesis. Hence we obtain $b_i \sim x_{i_1} t \sim x_{j_1} t = b_j$.

Case 3: If $t \in \mathcal{O}$ and $x_{i_1} t \in \mathcal{O}$, then $x_{j_1} x_{i_1} t = x_{i_1} b_j \in \partial \mathcal{O}$ is a next-door neighbor of b_j. Since $\gcd(x_{i_1} b_j, b_i) = x_{i_1} g$, we have $x_{i_1} b_j \sim b_i$ by induction hypothesis. Hence we obtain $b_i \sim x_{i_1} b_j \sim b_j$. $\square$

Let $\{e_1, \ldots, e_\nu\}$ be the canonical basis of the free module P^ν. The **fundamental syzygies** $\sigma_{ij} = (\mathrm{lcm}(b_i, b_j)/b_i)\, e_i - (\mathrm{lcm}(b_i, b_j)/b_j)\, e_j$ generate the border syzygy module $\mathrm{Syz}_P(b_1, \ldots, b_\nu)$ (see, for instance, [4, Theorem 2.3.7.b]). We are going to show that there exists a much more efficient set of generators for this syzygy module.

Definition 20 Let $\mathcal{O}$ be an order ideal with border $\partial \mathcal{O} = \{b_1, \ldots, b_\nu\}$.

a) For two next-door neighbors b_i, b_j, i.e. for $b_i = x_k b_j$, the fundamental syzygy σ_{ij} has the form $\tau_{ij} = e_i - x_k e_j$ and is called a **next-door neighbor syzygy**.

b) For two across-the-street neighbors b_i, b_j, i.e. for $x_k b_i = x_\ell b_j$, the fundamental syzygy σ_{ij} has the form $\upsilon_{ij} = x_k e_i - x_\ell e_j$ and is called an **across-the-street neighbor syzygy**.

c) The set of all **neighbor syzygies** is the set of all next-door or across-the street neighbor syzygies.

Proposition 21 *The set of all neighbor syzygies generates* $\mathrm{Syz}_P(b_1, \ldots, b_\nu)$.

Proof. Since the module $\mathrm{Syz}_P(b_1, \ldots, b_\nu)$ is generated by the set of fundamental syzygies $\{\sigma_{ij} \mid 1 \le i < j \le \nu\}$, it suffices to show that every fundamental syzygy is a P-linear combination of neighbor syzygies. For notational convenience we let $\sigma_{ji} = -\sigma_{ij}$ for $1 \le i < j \le n$. Let $b_{k_0}, \ldots, b_{k_s}$ be a sequence of border terms constructed in the proof of Proposition 19, i.e. such that $b_{k_0} = b_i$, $b_{k_s} = b_j$ and $b_{k_{\ell-1}}, b_{k_\ell}$ are neighbors for $\ell = 1, \ldots, s$. We claim that there are terms $f_1, \ldots, f_s \in \mathbb{T}^n$ such that $\sigma_{ij} = \sum_{\ell=1}^{s} f_\ell \varphi_\ell$, where φ_ℓ is the neighbor syzygy between $b_{k_{\ell-1}}$ and b_{k_ℓ}.

To prove this claim, we proceed by induction on s. For $s=1$, the terms b_i and b_j are neighbors and σ_{ij} is the corresponding neighbor syzygy. For $s>1$, let $b_i = x_{i_1}^{\alpha_1}\cdots x_{i_p}^{\alpha_p}\cdot \gcd(b_i,b_j)$ and $b_j = x_{j_1}^{\beta_1}\cdots x_{j_q}^{\beta_q}\cdot \gcd(b_i,b_j)$ as in the proof of Proposition 19. If $q=0$, i.e. if $b_i = x_{i_1}^{\alpha_1}\cdots x_{i_p}^{\alpha_p}\cdot b_j$, then $b_i = b_{k_0} = x_{i_p} b_{k_1}$ and therefore $\sigma_{ij} - t_{ij}\tau_{k_0 k_1} = t_{ij} x_{i_p} e_{k_1} - t_{ji} e_{k_s} = x_{i_p} \sigma k_1 k_s$ is a syzygy of b_{k_1} and b_{k_s}. The claim follows by induction. If $q \ge 1$, we write $b_j = x_{j_1} t$ with $t \in \mathbb{T}^n$ and check the same three cases as in the proof of Proposition 19.

Case 1: If $t \in \partial\mathcal{O}$, then $b_{k_{s-1}} = t$ and $\tau_{k_s k_{s-1}} = e_{k_s} - x_{j_1} e_{k_{s-1}}$ is a next-door neighbor syzygy. Thus $\sigma_{ij} + t_{ji}\tau_{k_s k_{s-1}} = t_{ij} e_{k_0} - t_{ji} x_{j_1} e_{k_{s-1}} = x_{j_1}\sigma_{k_0 k_{s-1}}$ is a syzygy of b_{k_0} and $b_{k_{s-1}}$. The claim follows by induction.

Case 2: If $t \in \mathcal{O}$ and $b_{k_{s-1}} = x_{i_1} t \in \partial\mathcal{O}$, then $x_{i_1} b_{k_s} = x_{i_1} x_{j_1} t = x_{j_1} b_{k_{s-1}}$ and $\upsilon_{k_s k_{s-1}} = x_{i_1} e_{k_s} - x_{j_1} e_{k_{s-1}}$ is an across-the-street neighbor syzygy. Since $t_{k_s k_0} = \operatorname{lcm}(b_{k_s}, b_{k_0})/b_{k_s} = x_{i_1}^{\alpha_1}\cdots x_{i_p}^{\alpha_p}$, and since $x_{i_1} \mid b_{k_{s-1}}$ implies $\alpha_{i_1} \ge 1$, we have a factorization $t_{k_s k_0} = x_{i_1}\cdot t'$ with $t' \in \mathbb{T}^n$. Then $\sigma_{ij} + t'\upsilon_{k_s k_{s-1}} = t_{ij} e_{k_0} - t' x_{j_1} e_{k_{s-1}} = x_{j_1}\sigma_{k_0 k_{s-1}}$ is a syzygy of b_{k_0} and $b_{k_{s-1}}$. The claim follows by induction.

Case 3: If $x_{i_1} t \in \mathcal{O}$ and $b_{k_{s-1}} = x_{i_1} b_j \in \partial\mathcal{O}$, then $\tau_{k_{s-1} k_s} = e_{k_{s-1}} - x_{i_1} e_{k_s}$ is a next-door neighbor syzygy. Again we write $t_{k_s k_0} = x_{i_1} t'$ with $t' \in \mathbb{T}^n$ and compute $\sigma_{ij} - t'\tau_{k_{s-1} k_s} = t_{ij} e_{k_0} - t' e_{k_{s-1}} = \sigma_{k_0 k_{s-1}}$. Again, the claim follows by induction. $\square$

Notice that neighbor syzygies are particularly simple: They are binomials, and their coefficients are either the constants ± 1 or indeterminates. Now we apply our knowledge of border syzygies to characterize border bases via the lifting of syzygies; this corresponds to [4, Proposition 2.3.12]. We are especially interested in the syzygy module

$$\operatorname{Syz}(g_1,\dots,g_\nu) \;=\; \{(f_1,\dots,f_\nu) \in P^\nu \mid f_1 g_1 + \cdots + f_\nu g_\nu = 0\}$$

and begin with defining a lifting of a border syzygy.

Definition 22 Let $(f_1,\dots,f_\nu) \in \operatorname{Syz}_P(b_1,\dots,b_\nu)$ be a border syzygy. A syzygy $(F_1,\dots,F_\nu) \in \operatorname{Syz}_P(g_1,\dots,g_\nu)$ is a **lifting** of $(f_1,\dots,f_\nu)$ if one of the following two cases occurs:

1. $\sum_{j=1}^{\nu} f_j g_j = 0$ and $F_i - f_i = 0$ for all $i \in \{1,\dots,\nu\}$, or
2. $\sum_{j=1}^{\nu} f_j g_j \ne 0$ and for all $i \in \{1,\dots,\nu\}$ such that $F_i - f_i \ne 0$ we have $\deg(F_i - f_i) \le \operatorname{ind}_{\mathcal{O}}(\sum_{j=1}^{\nu} f_j g_j) - 1$.

In the second case the combination $\sum_j (F_j - f_j) g_j$ cancels $\sum_j f_j g_j$, and the degree bound insures that there are no internal cancellations in the combination $\sum_j (F_j - f_j) g_j$ beyond the index of $\sum_j f_j g_j$. The next example shows that there are liftings and, moreover, that liftings of neighbor syzygies are very easy to compute if G is a border basis.

Example 23 Assume that G is an $\mathcal{O}$-border basis.

a) Given a next-door neighbor syzygy $\tau_{ij} = e_i - x_k e_j$, all terms appearing in $g_i - x_k g_j$ have index ≤ 1. Therefore there exist $c_1, \ldots, c_m \in K$ such that the support of $g_i - x_k g_j - \sum_{m=1}^{\nu} c_m g_m$ is contained in $\mathcal{O}$. Since G is a border basis, it follows that $\varphi_{ij} = e_i - x_k e_j - \sum_{m=1}^{\nu} c_m e_m$ is a syzygy of $(g_1, \ldots, g_\nu)$. This syzygy lifts τ_{ij}, because $g_i - x_k g_j = 0$ or $\deg(c_m) = 0 < 1 = \mathrm{ind}_{\mathcal{O}}(g_i - x_k g_j)$

b) Given an across-the-street neighbor syzygy $\upsilon_{ij} = x_k e_i - x_\ell e_j$, the only terms of index two appearing in $x_k g_i - x_\ell g_j$ are $x_k b_i$ and $x_\ell b_j$. Since these two terms cancel and all other terms have index ≤ 1, there exist $d_1, \ldots, d_\nu \in K$ such that the support of $x_k g_i - x_\ell g_j - \sum_{m=1}^{\nu} d_m g_m$ is contained in $\mathcal{O}$. Again the border basis property of G implies that $\psi_{ij} = x_k e_i - x_\ell e_j - \sum_{m=1}^{\nu} d_m e_m$ is a syzygy of $(g_1, \ldots, g_\nu)$ which lifts υ_{ij}, because $x_k g_i - x_\ell g_j = 0$ or $\deg(d_m) = 0 < 1 = \mathrm{ind}_{\mathcal{O}}(x_k g_i - x_\ell g_j)$.

Since the index need not be monotone with respect to multiplication by a term, the index of $\sum_{j=1}^{\nu} f_j g_j$ can actually be larger than the index r of the terms in $f_1 b_1 + \cdots + f_\nu b_\nu$. The following example is a case in point.

Example 24 Let $\mathcal{O} = \{1, x, x^2\} \subset \mathbb{T}^2$. Then $\partial\mathcal{O} = \{y, xy, x^2 y, x^3\}$. The set $G = \{g_1, g_2, g_3, g_4\}$ where $g_1 = y - x^2$, $g_2 = xy$, $g_3 = x^2 y$, and $g_4 = x^3$ is an $\mathcal{O}$-border basis of $I = (g_1, g_2, g_3, g_4)$. We have $\mathrm{ind}_{\mathcal{O}}(f b_1) = 1$ for $f = x^2$ and $b_1 = y$, while the polynomial $f g_1 = x^2 y - x^4$ has $\mathcal{O}$-index two.

The next proposition is the main result of this section. It characterizes border bases via liftings of pure border syzygies.

Proposition 25 (Border Bases and Liftings of Border Syzygies)
In the prescribed setting, the set G is an $\mathcal{O}$-border basis of I if and only if one of the following equivalent conditions is satisfied.

E_1. *Every border syzygy lifts to a syzygy of $(g_1, \ldots, g_\nu)$.*

E_2. *Every neighbor syzygy lifts to a syzygy of $(g_1, \ldots, g_\nu)$.*

Proof. First we show that a border basis satisfies E_1. Let $(f_1, \ldots, f_\nu)$ be a border syzygy, and let $f = f_1 g_1 + \cdots + f_\nu g_\nu$. Using the Border Division Algorithm, we compute a representation $f = h_1 g_1 + \cdots + h_\nu g_\nu$ with $\deg(h_i) \leq$

$\mathrm{ind}_{\mathcal{O}}(f) - 1$. The normal remainder is zero, since $f \in I$ and G is a border basis of I. Now $(f_1 - h_1, \dots, f_\nu - h_\nu)$ is a syzygy of $(g_1, \dots, g_\nu)$ that lifts $(f_1, \dots, f_\nu)$.

Since E_1 logically implies E_2, it remains to prove that G is a border basis if E_2 holds true. Given a next-door neighbor syzygy $\tau_{ij} = e_i - x_k e_j$, we have $g_i - x_k g_j = 0$ or the index of $g_i - x_k g_j$ is one. Therefore any lifting of τ_{ij} has the form $\tau_{ij} - \sum_{m=1}^{\nu} c_m e_m$ with $c_m \in K$. Given an across-the street neighbor syzygy $\upsilon_{ij} = x_k e_i - x_\ell e_j$, the polynomial $x_k g_i - x_\ell g_j$ is zero or its index is one. Therefore any lifting of υ_{ij} has the form $\upsilon_{ij} - \sum_{m=1}^{\nu} c_m e_m$ with $c_m \in K$. In both cases the S-polynomial has the shape $\mathrm{S}(g_i, g_j) = \sum_{m=1}^{\nu} c_m g_m$, and the claim follows from the last part of the proof of Proposition 18. $\square$

There is an important difference between liftings in border basis and those in Gröbner basis theory: condition E_1 guarantees liftings for all border syzygies, whereas in Gröbner basis theory we can only lift homogeneous syzygies of leading terms. To examine which kind of border syzygies is the correct analogue of homogeneous syzygies of leading terms, we introduce two particularly nice kinds of border syzygies.

Definition 26 Let $\mathcal{O}$ be an order ideal with border $\partial\mathcal{O} = \{b_1, \dots, b_\nu\}$, and let $k \in \mathbb{N}$.

a) A border syzygy $(f_1, \dots, f_\nu)$ is called **pure of index** k if $\bigcup_{i=1}^{\nu} \mathrm{Supp}(f_i b_i)$ is contained in $\partial^k\mathcal{O}$. It is called **pure** if it is pure of some index.
b) A border syzygy $(f_1, \dots, f_\nu)$ is called **perfect of index** k if it is pure of index k and the polynomials f_i are homogeneous of degree $k-1$.

The following example amplifies the details of this definition.

Example 27

a) The next-door neighbor syzygy $\tau_{ij} = e_i - x_r e_j$ is a pure syzygy of index one. We note that τ_{ij} is not perfect of index one, since $\deg(1) \neq \deg(x_r)$.
b) The across-the-street neighbor syzygy $\upsilon_{ij} = x_r e_i - x_s e_j$ looks like a perfect syzygy. It is indeed perfect of index two, if $x_r b_i \notin \partial\mathcal{O}$. However, if $x_r b_i \in \partial\mathcal{O}$, then it is not a perfect syzygy: We have $\mathrm{ind}_{\mathcal{O}}(x_r b_i) = \mathrm{ind}_{\mathcal{O}}(x_r b_j) = 1$ and $\deg(x_r) = \deg(x_s) = 1$; the index and the degree fail the perfectness condition $\mathrm{ind} = \deg + 1$.
c) Let $\mathcal{O} = \{1, x, y\}$ with border $\partial\mathcal{O} = \{x^2, xy, y^2\}$. Then $(y, -x-y, x)$ is a perfect syzygy of index two. Unlike the analogous situation in Gröbner basis theory, perfect border syzygies can have components with several terms in their support.

Since neighbor syzygies are pure, a border prebasis G is an $\mathcal{O}$-border basis of I if and only if the following condition is satisfied:

E_3. Every pure border syzygy lifts to a syzygy of $(g_1, \ldots, g_\nu)$.

Pure border syzygies play a role analogous to homogeneous syzygies of the tuple of leading terms in Gröbner basis theory.

Remark 28 Every syzygy $s = (f_1, \ldots, f_\nu) \in \mathrm{Syz}_P(b_1, \ldots, b_\nu)$ decomposes into pure syzygies, $s = \sum_{r \geq 1} s_r$, where s_r collects all terms in the support of s such that the corresponding summand in $f_1 b_1 + \cdots + f_\nu b_\nu$ has index r. Since $P = \bigoplus_{i \geq 0} (\bigoplus_{\{t \mid \mathrm{ind}_{\mathcal{O}}(t) = i\}} K \cdot t)$, the tuples s_r are again syzygies of $(b_1, \ldots, b_\nu)$. Therefore, the characterization of border bases via liftings restricts to pure syzygies.

Can we restrict our characterization via liftings to the even simpler perfect syzygies? If G is a border bases, then E_1 implies that every perfect border syzygy possesses a lifting in $\mathrm{Syz}_P(g_1, \ldots, g_\nu)$. However, the converse is not true in general, as our next example shows.

Example 29 Let $\mathcal{O} = \{1, x\} \subset \mathbb{T}^2$. Then $\partial\mathcal{O} = \{y, xy, x^2\}$, and every perfect border syzygy $f_1 y + f_2 xy + f_3 x^2 = 0$ is of the form $(f_1, f_2, f_3) = (0, fx, -fy)$ with a homogeneous polynomial $f \in K[x, y]$. The $\mathcal{O}$-border prebasis $G = \{g_1, g_2, g_3\}$ with $g_1 = y - 1$, $g_2 = xy$, and $g_3 = x^2$, is not a border basis, because $x = g_2 - x g_1 \in I$. However, every perfect border syzygy $(0, fx, -fy)$ is its own lifting.

In Example 23 we lifted every next-door neighbor syzygy τ_{ij} to φ_{ij} and every across-the-street neighbor syzygy υ_{ij} to ψ_{ij}.

Conjecture. The liftings φ_{ij}, ψ_{ij} of the neighbor syzygies generate the syzygy module $\mathrm{Syz}_P(g_1, \ldots, g_\nu)$ of a border basis $G = \{g_1, \ldots, g_\nu\}$.

This conjecture is motivated by the analogy with [4, Proposition 3.1.4] and supported by several examples that we computed with CoCoA. As a further indication, we show that the liftings φ_{ij}, ψ_{ij} generate at least the following special syzygies of $(g_1, \ldots, g_\nu)$.

Proposition 30 *Let $G = \{g_1, \ldots, g_\nu\}$ be an $\mathcal{O}$-border basis. Every syzygy $(f_1, \ldots, f_\nu) \in \mathrm{Syz}_P(g_1, \ldots, g_\nu)$ such that $f_1 b_1 + \cdots + f_\nu b_\nu = 0$ is contained in $\langle \{\varphi_{ij} \mid b_i, b_j$ next-door neighbors$\} \cup \{\psi_{ij} \mid b_i, b_j$ across-the-street neighbors$\} \rangle$.*

Proof. Since $(f_1, \ldots, f_\nu)$ is a border syzygy, there exist polynomials $h_{ij}, k_{ij} \in P$ such that $(f_1, \ldots, f_\nu) = \sum_{i,j} h_{ij} \tau_{ij} + \sum_{i,j} k_{ij} \upsilon_{ij}$ where the sums range over all i, j such that b_i, b_j are next-door or across-the-street neighbors, respectively and such that each unordered pair of neighbors appears only once. For these indices, we write $\varphi_{ij} = \tau_{ij} - \sum_{m=1}^{\nu} c_m^{(i,j)} e_m$ and $\psi_{ij} = \upsilon_{ij} - \sum_{m=1}^{\nu} d_m^{(i,j)} e_m$ with $c_m^{(i,j)}, d_m^{(i,j)} \in K$. Then we calculate

$$\begin{aligned}
0 = f_1 g_1 + \cdots + f_\nu g_\nu &= \sum_{i,j} h_{ij} \tau_{ij}(g_1, \ldots, g_\nu) + \sum_{i,j} k_{ij} \upsilon_{ij}(g_1, \ldots, g_\nu) \\
&= \sum_{i,j} h_{ij}(\varphi_{ij} + \sum_{m=1}^{\nu} c_m^{(i,j)} e_m)(g_1, \ldots, g_\nu) + \sum_{i,j} k_{ij}(\psi_{ij} + \sum_{m=1}^{\nu} d_m^{(i,j)} e_m)(g_1, \ldots, g_\nu) \\
&= \sum_{m=1}^{\nu} (\sum_{i,j} c_m^{(i,j)} + \sum_{i,j} d_m^{(i,j)})\, g_m.
\end{aligned}$$

In the last step the other summands disappear, because they involve the liftings φ_{ij} and ψ_{ij}, i.e. syzygies of $(g_1, \ldots, g_\nu)$. Since $\{g_1, \ldots, g_\nu\}$ is K-linearly independent, we get $\sum_{i,j} c_m^{(i,j)} + \sum_{i,j} d_m^{(i,j)} = 0$ for every $m \in \{1, \ldots, \nu\}$, and, therefore,

$$\begin{aligned}
(f_1, \ldots, f_\nu) &= \sum_{i,j} h_{ij}(\varphi_{ij} + \sum_{m=1}^{\nu} c_m^{(i,j)} e_m) + \sum_{i,j} k_{ij}(\psi_{ij} + \sum_{i,j} d_m^{(i,j)} e_m) \\
&= \sum_{i,j} h_{ij} \varphi_{ij} + \sum_{i,j} k_{ij} \psi_{ij}
\end{aligned}$$

is contained in the module generated by the syzgyies φ_{ij}, ψ_{ij}. $\square$

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