Some Thoughts about Border Bases

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There are things that are so serious
that you can only joke about them.

Approximate
Border
Bases
Or
Traditional
Theorems?

Introduction

In practical applications good models lead most of the time to zero-dimensional ideals. The reason is that a zero-dimensional ideal is an algebraic structure which encodes a system of polynomial equations with only a **finite number of solutions**.

In pure mathematics the coefficients of the equations are treated according to their formal properties. Consequently, mathematicians like exact numbers, even better if they are elements of a field. But in practice one has to cope with **approximate data** which lead to phenomena of **instability**.

After the fundamental work of Buchberger, **Gröbner bases** gained the status of the *most known and used algebraic tool*. However, they do not behave always well with respect to stability issues. More recently, after the fundamental work of Stetter, **border bases** started to be considered as better suited algebraic tools for several practical problems.

How did we (i.e. the CoCoA Team) get interested in border bases?

- Statistics – A few year ago, I and Massimo Caboara were able to use Border Bases as a theoretical tool, and gave a complete solution to a problem of **Design of Experiments**, a branch of Statistics (see [CR01]).

- SHELL – The main idea of the CoCoA-SHELL collaborative project is to use **polynomial models** for improving **oil exploration and recovery**. At this early stage, what is most needed is **computer algebra** software offering extremely high efficiency in performing basic operations with polynomial objects such as ideals, modules, and syzygies.

- CoCoA – Currently, there is only a single computer algebra software library with such capabilities: the one being developed as **part of the CoCoA 5** project. And we are including Border Bases into the library.

- The book(s) – In the last few (?) years I and Martin worked on **the Book** and included a lot of material on Gröbner and Border Bases.

Part I: Deformations of Border Bases

Recently **border bases** started to be considered as useful algebraic tools. Let us see why by asking a few **questions** and giving some **answers**.

Q1. What are Border Bases (BB)?

Q2. How do they generalize Gröbner Bases (GB)?

Q3. How do we compute BB?

Q4. Are BB numerically stable?

– The answers to Q1, Q2 are given in [KR05], and will be recalled using examples.

– The answer to Q3 is given in the recent paper [KK05] and in some work of Mourrain and coworkers and will be treated by Kreuzer, Abbott and Poulisse.

Let us make some remarks on Q4. To start with, let us look at an example.

Example This is Example 6.4.1 of [KR05].

Consider the polynomial system

$$\begin{array}{rcccl} f_1 & = & \frac{1}{4}\,x^2 + y^2 - 1 & = & 0 \\ f_2 & = & x^2 + \frac{1}{4}\,y^2 - 1 & = & 0 \end{array}$$

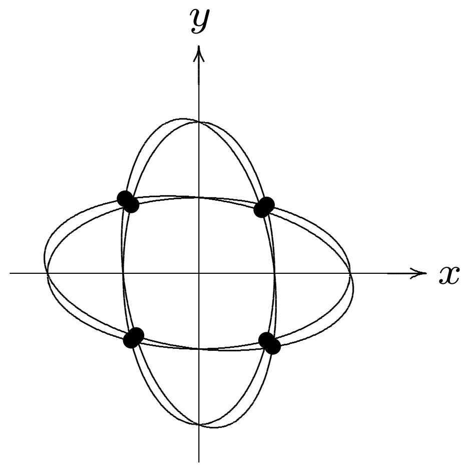
The intersection of $\mathcal{Z}(f_1)$ and $\mathcal{Z}(f_2)$ in $\mathbb{A}^2(\mathbb{C})$ consists of the four points $\mathbb{X} = \{(\pm\sqrt{4/5},\, \pm\sqrt{4/5})\}$.

Using Gröbner basis theory, we describe this situation as follows. The set $\{x^2 - \frac{4}{5},\, y^2 - \frac{4}{5}\}$ is the reduced Gröbner basis of the ideal $I = (f_1, f_2) \subseteq \mathbb{C}[x, y]$ with respect to $\sigma =$`DegRevLex`. Therefore we have $\mathrm{LT}_\sigma(I) = (x^2,\, y^2)$, and the residue classes of the terms in $\mathbb{T}^2 \setminus \mathrm{LT}_\sigma\{I\} = \{1,\, x,\, y,\, xy\}$ form a $\mathbb{C}$-vector space basis of $\mathbb{C}[x, y]/I$.

Now consider the slightly perturbed polynomial system

$$\begin{array}{rcccl} \tilde{f}_1 & = & \frac{1}{4}\,x^2 + y^2 + \varepsilon\,xy - 1 & = & 0 \\ \tilde{f}_2 & = & x^2 + \frac{1}{4}\,y^2 + \varepsilon\,xy - 1 & = & 0 \end{array}$$

where ε is a small number. The intersection of $\mathcal{Z}(\tilde{f}_1)$ and $\mathcal{Z}(\tilde{f}_2)$ consists of four perturbed points $\widetilde{\mathbb{X}}$ close to those in $\mathbb{X}$.



Now the ideal $\tilde{I} = (\tilde{f}_1, \tilde{f}_2)$ has the reduced σ-Gröbner basis

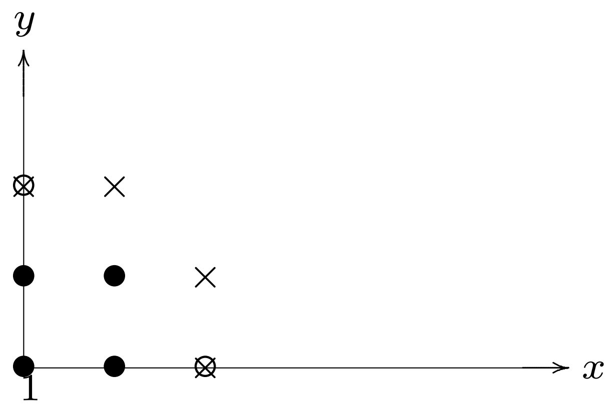
$$\{x^2 - y^2,\, xy + \tfrac{5}{4\varepsilon}\, y^2 - \tfrac{1}{\varepsilon},\, y^3 - \tfrac{16\varepsilon}{16\varepsilon^2 - 25}\, x + \tfrac{20}{16\varepsilon^2 - 25}\, y\}$$

Moreover, we have $\mathrm{LT}_\sigma(\tilde{I}) = (x^2,\, xy,\, y^3)$ and $\mathbb{T}^2 \setminus \mathrm{LT}_\sigma\{\tilde{I}\} = \{1,\, x,\, y,\, y^2\}$.

A **small** change in the coefficients of f_1 and f_2 has led to a **big** change in the Gröbner basis of (f_1, f_2) and in the associated vector space basis of $\mathbb{C}[x, y]/(f_1, f_2)$, although the zeros of the system have not changed much. In numerical analysis this kind of unstable behaviour is called a *representation singularity*.

Let us reconsider the above Example.

Let $P = \mathbb{Q}[x,y]$, let $I = (\frac{1}{4}\,x^2 + y^2 - 1,\, x^2 + \frac{1}{4}\,y^2 - 1)$, and let $\tilde{I} = (\frac{1}{4}\,x^2 + y^2 + \varepsilon\,xy - 1,\, x^2 + \frac{1}{4}\,y^2 + \varepsilon\,xy - 1)$ with a small number ε. Then both I and $\tilde{I}$ have a border basis with respect to $\mathcal{O} = \{1, x, y, xy\}$.



The border of $\mathcal{O}$ is $\partial\mathcal{O} = \{x^2, x^2y, xy^2, y^2\}$. The $\mathcal{O}$-border basis of I is

$$\{x^2 - \tfrac{4}{5},\ x^2y - \tfrac{4}{5}y,\ xy^2 - \tfrac{4}{5}x,\ y^2 - \tfrac{4}{5}\}$$

The $\mathcal{O}$-border basis of $\tilde{I}$ is

$$\{x^2 + \tfrac{4}{5}\,\varepsilon xy - \tfrac{4}{5},\ x^2y - \tfrac{16\varepsilon}{16\varepsilon^2 - 25}\,x + \tfrac{20}{16\varepsilon^2 - 25}\,y,$$
$$xy^2 + \tfrac{20}{16\varepsilon^2 - 25}\,x - \tfrac{16\varepsilon}{16\varepsilon^2 - 25}\,y,\ y^2 + \tfrac{4}{5}\,\varepsilon xy - \tfrac{4}{5}\}$$

When we vary the coefficients of xy in the two generators from zero to ε, we see that one border basis changes continuously into the other. Thus the border basis is numerically stable under a small perturbations of the coefficient of xy.

Let us now try to go on with the help of some experiments

Let us consider another **perturbed** ideal I_ε generated by the following set of **perturbed equations**

$$(1/4-\varepsilon)x^2+\varepsilon xy+(1-\varepsilon)y^2-1-\varepsilon, \quad x^2+1/4y^2-1-\varepsilon$$

It yields the following border basis when $\varepsilon = 10^{-6}$

$$\begin{aligned}
&x^2 \quad - \tfrac{1}{374999997}xy - \tfrac{7499999974999999}{9374999925000000},\\
&x^2y - \tfrac{117187502734375015625 0}{10986327949218751}x - \tfrac{28124999681249997000000003}{35156249437500003250000000}y,\\
&y^2 \quad + \tfrac{4}{374999997}xy - \tfrac{7500000175000001}{9374999925000000},\\
&xy^2 - \tfrac{28125000431249998499999997}{35156249437500003250000000}x + \tfrac{7499999974999999}{87890623593750008125 0000}y
\end{aligned}$$

This differs from the border basis of I only slightly. The **variation** of the coefficients is of the **same order of magnitude** as ε itself.

What happens if, instead of perturbing the ideal, we perturb the coordinates of the points? If you try with CoCoA you see the *same phenomenon* happening.

Can we explain these experimental data?

The **generic border prebasis** is defined by "marking" the terms in $\partial\mathcal{O}$ and then letting the coefficients vary freely to get generic linear combinations of the elements of $\mathcal{O}$.

Let N denote the number of elements in the basis $\mathcal{O}$, and let B denote the number of elements in the **generic border prebasis**.

It is clear that the generic border prebasis is parametrized by an affine space of dimension $D = N \times B$. In our previous example, we have $N = 4$, $B = 4$, hence $D = 16$.

To get the **border basis variety** we need to impose that the multiplication matrices are **pairwise commuting**.

Q5. Let us denote by $\mathcal{BO}$ the $\mathcal{O}$-border basis variety associated to the order ideal $\mathcal{O} = [1,\ x,\ y,\ xy\ ]$. What is the *expected dimension* of $\mathcal{BO}$?

The first answer comes from CoCoA.We get $\dim(\mathcal{BO}) = 8$. But CoCoA is in trouble as soon as we enlarge the cardinality of $\mathcal{O}$ just a bit.

So, let us try to understand the number 8. Every perturbation of the set of four points is still a set of four points. A generic set of four points needs **8 parameters** to be represented.

Let us see it from another perspective. The ideal of a set of four points is the complete intersection of two quadratic equations. Now, every quadratic equation depends on 6 homogeneous parameters. So we have to count the dimension of 2-dimensional subspaces of $\mathbb{A}^6$ or, equivalently, the dimension of $\mathrm{Grass}(1,5)$. It turns out that its dimension is 8. In conclusion, we have

$$dim(\mathcal{BO}) = 8$$

Q6. How can $\mathcal{O}$ **not** be a basis (as a K-vector space) of the quotient ring P/I where I is the defining ideal of a set of four points?

We know that $\dim_K(P/I) = 4$. Therefore it is enough to impose that the elements in $\mathcal{O}$ are linearly independent. We conclude that

$\mathcal{O}$ is not a basis of P/I if the ideal I contains a polynomial which is a linear combination of the elements of $\mathcal{O}$.

Certainly this is the case if the four points are collinear. It is also the case for instance if $\mathbb{X} = \{p_1, p_2, p_3, p_4\}$ where $p_1 = (0,0)$, $p_2 = (0,1)$, $p_3 = (0,2)$, $p_4 = (1,0)$. In this case the ideal is $I(\mathbb{X}) = (xy,\ x^2 - x,\ y^3 - 3y^2 + 2y)$ and we see that xy is in the ideal, so that $\mathcal{O}$ is not linearly independent modulo $I(\mathbb{X})$.

Finally, a contribution to Q4.

Suppose we have an $\mathcal{O}$-border basis $\mathcal{B}$. The ideal I generated by $\mathcal{B}$ is such that $\dim_K(P/I) = 4$, so it is the ideal of a 0-dimensional scheme with multiplicity 4. **Most** of the ideals I of that type represent 4 distinct points, and $\mathcal{O}$ is a basis of the quotient vector spaces P/I. Since the elements of P can be interpreted as **functions** on the set of points, we can say that the functions $1,\ x,\ y,\ xy$ are linearly independent on the set. Suppose now we do one of the two following operations

a) We perturb the equations of a minimal set of generators of I.

b) We perturb the coordinates of the four points.

In both cases what we get is still a set of four points, and the four functions $1,\ x,\ y, xy$ are still linearly independent, since the determinant of the evaluation matrix was non-zero before, and hence it is still different from zero.

But **be careful**. This is a special case of a complete intersection of two conics, hence perturbing the two conics still yields two conics, and hence four points. If, instead we perturb another set of generators, for instance a border basis, the situation may change dramatically, in the sense that the perturbed ideal no longer represent a set of four points, but for instance the unit ideal. Nevertheless, even in this case we do have **almost commuting matrices**. The geometric interpretation is that we move slightly outside the variety $\mathcal{BO}$.

A partial conclusion after the above **experiments** is the following.

Q4. Are BB numerically stable?

Yes, they are, in the sense that perturbing the coordinates of a finite set of points, we **move inside** $\mathcal{BO}$. Perturbing the equations, we possibly move outside, but not too far away. In some cases, for geometric reasons, even perturbing the equations does not imply that we move outside $\mathcal{BO}$, as in the case of the four points and the minimal generating set (two conics).

Part II: Computation of Border Bases

K field, $P = K[x_1, \ldots, x_n]$, $I \subseteq P$ 0-dim. ideal, $\mathcal{O} = \{t_1, \ldots, t_\mu\}$ order ideal.

Basis Transformation Algorithm

The following algorithm checks whether $\mathcal{O}$ supports a border basis of I and, in the affirmative, computes the $\mathcal{O}$-border basis $\{g_1, \ldots, g_\nu\}$ of I.

(T1) Choose a term ordering σ and compute $\mathcal{O}_\sigma\{I\} := \mathbb{T}^n \setminus \mathrm{LT}_\sigma\{I\}$.

(T2) If $\#\mathcal{O}_\sigma\{I\} \neq \mu$ then return "$\mathcal{O}$ has the wrong cardinality" and stop.

(T3) Let $\mathcal{O}_\sigma\{I\} = \{s_1, \ldots, s_\mu\}$. For $1 \leq m \leq \mu$, compute the coefficients $\tau_{im} \in K$ of the normal form $\mathrm{NF}_{\sigma,I}(t_m) = \sum_{i=1}^{\mu} \tau_{im} s_i$. Let $\mathcal{T}$ be the matrix $(\tau_{im})_{1 \leq i,m \leq \mu}$.

(T4) If $\det \mathcal{T} = 0$ then return "$\mathcal{O}$ does not support a border basis of I" and stop.

(T5) Let $\partial\mathcal{O} = \{b_1, \ldots, b_\nu\}$. For $1 \leq j \leq \nu$, compute the coefficients $\beta_{ij} \in K$ of $\mathrm{NF}_{\sigma,I}(b_j) = \sum_{i=1}^{\mu} \beta_{ij} s_i$. Let $\mathcal{B}$ be the matrix $(\beta_{ij})_{1 \leq i \leq \mu, 1 \leq j \leq \nu}$.

(T6) Compute $(\alpha_{ij}) = \mathcal{T}^{-1}\mathcal{B}$. Return $g_j := b_j - \sum_{i=1}^{\mu} \alpha_{ij} t_i$ for $1 \leq j \leq \nu$.

Mourrain's Generic Algorithm

Let $\mathcal{F} := \{f_1, \ldots, f_s\} \subset P$ be a set of polynomials that generates a 0-dim. ideal $I = \langle \mathcal{F} \rangle_P$. The following algorithm computes a vector subspace $B \subseteq P$ connected to 1 such that $P = B \oplus I$.

(M1) Choose a finite-dim. vector subspace $U \subseteq P$ that is connected to 1 and contains $f_1, \ldots, f_s$.

(M2) Let $V_0 = \langle f_1 \ldots, f_s \rangle_K$. Let $\ell = 0$.

(M3) Compute $V_{\ell+1} := V_\ell^+ \cap U$, where $V_\ell^+ = V_\ell + x_1 V_\ell + \cdots + x_n V_\ell$.

(M4) If $V_{\ell+1} \neq V_\ell$ then increase ℓ by 1 and go to step (M3).

(M5) Compute a vector subspace B connected to 1 such that $U = B \oplus V_\ell$.

(M6) If $B^+ \not\subseteq U$ then replace U with U^+ and go to step 3. Otherwise return B and stop.

Problems: Make (M5) effective, produce $B = \langle \mathcal{O} \rangle$ with an order ideal $\mathcal{O}$.

Ideas:

1. U is the *universe* for our computations. All computations take place in a finite dimensional K-vector space.

2. The vector space $V_\infty = V_\ell$ for $\ell \gg 0$ is called the *stable U-span* of $f_1, \ldots, f_s$. It is an approximation of $I \cap U$. The ideal $\tilde{I} = \langle V_\infty \rangle$ is an approximation of I.

3. During the computation of V_∞ we can keep track of vector space bases such that the final basis $\{v_1, \ldots, v_r\}$ of V_∞ has pairwise different leading terms with respect to some term ordering σ. Then we have

$$\mathrm{LT}_\sigma(V_\infty) = \{\mathrm{LT}_\sigma(v_1), \ldots, \mathrm{LT}_\sigma(v_r)\}$$

4. If U is generated by an order ideal of monomials, then $\mathcal{O} = \mathrm{LT}_\sigma(U) \setminus \langle \mathrm{LT}_\sigma(v_1), \ldots, \mathrm{LT}_\sigma(v_r) \rangle$ is also an order ideal and $U = V_\infty \oplus \langle \mathcal{O} \rangle_K$.

5. If $\partial\mathcal{O} \subseteq U$ then the Buchberger Criterion for border bases shows that $\mathcal{O}$ supports a border basis of I.

The Border Basis Algorithm

Let $\mathcal{F} = \{f_1, \ldots, f_s\} \subset P$ generate a 0-dim. ideal $I = \langle \mathcal{F} \rangle_P$. Let σ be a degree-compatible term ordering. The following algorithm computes the $\mathcal{O}_\sigma\{I\}$-border basis $\{g_1, \ldots, g_\nu\}$.

(B1) Let $d := \max\{\deg(f_i) \mid 1 \leq i \leq s\}$ and $U := \langle \mathbb{T}^n_{\leq d} \rangle_K$.

(B2) Compute a basis $\mathcal{V} = \{v_1, \ldots, v_r\}$ of $\langle \mathcal{F} \rangle_K$ with $\mathrm{LT}_\sigma(v_i) \neq \mathrm{LT}_\sigma(v_j)$ for $i \neq j$.

(B3) Compute a basis extension $\mathcal{W}'$ for $\langle \mathcal{V} \rangle_K \subseteq \langle \mathcal{V}^+ \rangle_K$ so that the elements of $\mathcal{V} \cup \mathcal{W}'$ have pairwise different leading terms.

(B4) Let $\mathcal{W} = \{v_{r+1}, \ldots, v_{r+\varrho}\} = \{v \in \mathcal{W}' \mid \deg(v) \leq d\}$.

(B5) If $\varrho > 0$ then replace $\mathcal{V}$ with $\mathcal{V} \cup \mathcal{W}$, increase r by ϱ, and go to (B3).

(B6) Let $\mathcal{O} := \mathbb{T}^n_{\leq d} \setminus \{\mathrm{LT}_\sigma(v_1) \ldots \mathrm{LT}_\sigma(v_r)\}$.

(B7) If $\partial\mathcal{O} \not\subseteq U$ then increase d by one, update $U := \langle \mathbb{T}^n_{\leq d} \rangle_K$, and go to (B3).

(B8) Apply the Final Reduction Algorithm and return its result $(g_1, \ldots, g_\nu)$.

Example. Consider the polynomials
$f_1 = z^2 + 3y - 7z,\ f_2 = yz - 4y,\ f_3 = xz - 4y,\ f_4 = y^2 - 4y,\ f_5 = xy - 4y,$
$f_6 = x^5 - 8x^4 + 14x^3 + 8x^2 - 15x + 15y$ which generate the vanishing ideal of

$$\mathbb{X} = \{(-1,0,0),\ (0,0,0),\ (1,0,0),\ (3,0,0),\ (5,0,0),\ (4,4,4),\ (0,0,7)\} \subseteq \mathbb{Q}^3$$

Let $\sigma = \mathtt{DegRevLex}$.

1. The initial universe $U = \langle \mathbb{T}^3_{\leq 5} \rangle$ has dimension 56.

2. The computation of the stable span needs four basis extensions.

3. The result is $\mathcal{O} = \{1, x, x^2, x^3, x^4, y, z\}$ and $\partial\mathcal{O} \subset U$.

The Buchberger algorithm uses S-polynomials up to degree 6 (without criteria up to degree 7).

The **Improved Border Basis Algorithm** minimizes U at each step. At the beginning we start with the vector space U spanned by the divisors of $\bigcup_{i=1}^{s} \operatorname{Supp}(f_i)$. If we need to enlarge U, we only add $\partial\mathcal{O}$.
In this way the example is computed in a 19-dim. universe and needs only one basis extension.

Computation of Approximate Border Bases

Input: "Empirical" polynomials $f_1, \ldots, f_s \in P = \mathbb{R}[x_1, \ldots, x_n]$ with $s \geq n$.

Assumption: "Close by" there exists a 0-dimensional ideal $I \subset P$ with $\dim_K(P/I) \gg 0$.

Problem: Find I and compute a border basis!

Let $V \subset P$ be a vector space, let $f_1, \ldots, f_s$ be a basis of V, and let $\mathrm{Supp}(f_1) \cup \cdots \cup \mathrm{Supp}(f_r) = \{t_1, \ldots, t_r\}$.
Then V can be represented by the matrix

$$\mathcal{A} = \begin{pmatrix} a_{11} & \cdots & a_{1s} \\ \vdots & & \vdots \\ a_{r1} & \cdots & a_{rs} \end{pmatrix}$$

where $f_j = a_{1j}t_1 + \cdots + a_{rj}t_j$ and $a_{ij} \in K$.

Idea: Use the *Singular Value Decomposition* of $\mathcal{A}$ to filter out the polynomials in V which are "almost zero".

Here the "size" of a polynomial is measured by the Euclidean norm of its coefficient vector.

The Singular Value Decomposition (SVD)

Theorem. Given $\mathcal{A} \in \mathrm{Mat}_{m,n}(\mathbb{R})$, there exist orthogonal matrices $\mathcal{U} \in \mathrm{Mat}_{m,m}(\mathbb{R})$ and $\mathcal{V} \in \mathrm{Mat}_{n,n}(\mathbb{R})$ and a matrix $\mathcal{S} \in \mathrm{Mat}_{m,n}(\mathbb{R})$ such that

$$\mathcal{A} = \mathcal{U}\,\mathcal{S}\,\mathcal{V}^{\mathrm{tr}} = \mathcal{U} \cdot \begin{pmatrix} \mathcal{D} & 0 \\ 0 & 0 \end{pmatrix} \cdot \mathcal{V}^{\mathrm{tr}}$$

where $\mathcal{D} = \mathrm{diag}(\sigma_1, \ldots, \sigma_r)$ is a diagonal matrix such that $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$ and $r = \mathrm{rank}(\mathcal{A})$.

The numbers $\sigma_1, \ldots, \sigma_r$ are called the **singular values** of $\mathcal{A}$ and are uniquely determined by $\mathcal{A}$.

Here the matrices $\mathcal{U}$ and $\mathcal{V}$ have the following interpretation:

first r columns of $\mathcal{U}$	$\equiv$	column space of $\mathcal{A}$
last $m-r$ columns of $\mathcal{U}$	$\equiv$	kernel of $\mathcal{A}^{\mathrm{tr}}$
first r columns of $\mathcal{V}$	$\equiv$	row space of $\mathcal{A}$
last $n-r$ columns of $\mathcal{V}$	$\equiv$	kernel of $\mathcal{A}$

The Basis Extension Version of SVD

Let $\mathcal{A} \in \mathrm{Mat}_{m,n}(\mathbb{R})$, and let $\mathcal{A} = \mathcal{U}\,\mathcal{S}\,\mathcal{V}^{\mathrm{tr}}$ be its singular value decomposition. Let $\mathcal{B} \in \mathrm{Mat}_{m,n'}(\mathbb{R})$ be a further matrix. Then there exists an SVD of the combined matrix $(\mathcal{A} \mid \mathcal{B})$ of the form

$$(\mathcal{A} \mid \mathcal{B}) = \mathcal{U}' \cdot \mathcal{S}' \cdot (\mathcal{V}')^{\mathrm{tr}}$$

where the first r columns of $\mathcal{U}$ are columns of $\mathcal{U}' \in \mathrm{Mat}_{m,m}(\mathbb{R})$ and where $\mathcal{V}' \in \mathrm{Mat}_{n,n}(\mathbb{R})$ is of the form

$$\mathcal{V}' = \begin{pmatrix} \mathcal{V} & 0 \\ 0 & \mathcal{V}'' \end{pmatrix}$$

Application: Numerically Stable Version of V^+

Let $V = Kf_1 \oplus \cdots \oplus Kf_r \subset P$ be a vector space of polynomials and $\mathcal{A} = (a_{ij})$ its representing matrix. Let $\varepsilon > 0$ be the desired numerical precision.

1. We compute the SVD of $\mathcal{A}$ and get $\mathcal{A} = \mathcal{U}\,\mathcal{S}\,\mathcal{V}^{\mathrm{tr}}$.

2. Replace all singular values $\sigma_i < \varepsilon$ by zero. The resulting matrix $\widetilde{\mathcal{A}} = \mathcal{U}\,\widetilde{\mathcal{S}}\,\mathcal{V}^{\mathrm{tr}}$ represents the polynomial vector space V_{app} of smallest rank which is "close to" V.

3. The representing matrix of $(V_{\mathrm{app}})^+ = V_{\mathrm{app}} + x_1 V_{\mathrm{app}} + \cdots + x_n V_{\mathrm{app}}$ is of the form $(\widetilde{\mathcal{A}} \mid \mathcal{B})$. We use the basis extension version of SVD and get $(\widetilde{\mathcal{A}} \mid \mathcal{B}) = \mathcal{U}' \cdot \mathcal{S}' \cdot (\mathcal{V}')^{\mathrm{tr}}$ as above.

4. Again we replace the "new" singular values $< \varepsilon$ by zero. The resulting matrix represents the polynomial vector space V_{app}^+ of smallest rank which is "close to" V^+.

Notice that we have also computed an orthogonal basis of V_{app} and an orthogonal basis of V_{app}^+ which extends it.

Approximate Leading Term Sets

In the Border Basis Algorithm we need a vector space basis $\mathcal{V}$ of V with pairwise different leading terms and a vector space basis extension $\mathcal{V} \cup \mathcal{W}'$ for $V \subseteq V^+$ with pairwise different leading terms.

Given a finite dimensional vector space of empirical polynomials $V \subset P$, we may apply the SVD and assume that $V = V_{\mathrm{app}}$.

Let $\mathcal{A} = (a_{ij}) \in \mathrm{Mat}_{r,s}(\mathbb{R})$ be a matrix representing V. By choosing the ONB $\{f_1, \ldots, f_s\}$ of V provided by the SVD, we may assume that $r \geq s$ and $\mathcal{A} = \mathcal{U}\,\mathcal{D}$ where $\mathcal{U}$ represents $(f_1, \ldots, f_s)$.

Recall that the rows of $\mathcal{U}$ are indexed by the terms in $\mathrm{Supp}(V)$. We order the terms such that larger terms w.r.t. σ correspond to higher rows. The first entry $> \varepsilon$ in each column is the *approximate leading coefficient* of f_i, and the term indexing its row is the *approximate leading term* $\mathrm{LT}_\sigma^{\mathrm{app}}(f_i)$.

We bring $\mathcal{A}^{\mathrm{tr}}$ into row echelon form, where we allow only pivots $> \varepsilon$. We make sure that the resulting basis $(f'_1, \ldots, f'_r)$ of V satisfies $\|f'_i\| = 1$. Then the set

$$\mathrm{LT}_\sigma^{\mathrm{app}}(V) = \{\mathrm{LT}_\sigma^{\mathrm{app}}(f_1), \ldots, \mathrm{LT}_\sigma^{\mathrm{app}}(f_r)\}$$

is the *approximate leading term set* of V and $\{f'_1, \ldots, f'_r\}$ is a basis of V having pairwise different approximate leading terms.

The Approximate Border Basis Algorithm

Let $\{f_1, \ldots, f_s\} \subset P$ be a linearly independent set of $s \geq n$ empirical polynomials and $V = \langle f_1, \ldots, f_s \rangle_K$. Let σ be a degree-compatible term ordering. The following algorithm computes the $\mathcal{O}_\sigma\{I\}$-border basis $\{g_1, \ldots, g_\nu\}$ of an ideal $I = (g_1, \ldots, g_\nu)$ such that $f_1, \ldots, f_s$ are "close to" I.

(B1) Let $d := \max\{\deg(f_i) \mid 1 \leq i \leq s\}$ and $U := \langle \mathbb{T}^n_{\leq d} \rangle_K$.

(B2) Using the SVD, compute V_{app} and a unitary basis $\mathcal{V} = \{v_1, \ldots, v_r\}$ of V_{app} having pairwise different approximate leading terms.

(B3) Compute a unitary basis extension $\mathcal{W}'$ for $V_{\mathrm{app}} \subseteq V^+_{\mathrm{app}}$ so that the elements of $\mathcal{V} \cup \mathcal{W}'$ have pairwise different approximate leading terms.

(B4) Let $\mathcal{W} = \{v_{r+1}, \ldots, v_{r+\varrho}\} = \{v \in \mathcal{W}' \mid \deg(v) \leq d\}$.

(B5) If $\varrho > 0$ then replace $\mathcal{V}$ with $\mathcal{V} \cup \mathcal{W}$, increase r by ϱ, and go to (B3).

(B6) Let $\mathcal{O} := \mathbb{T}^n_{\leq d} \setminus \{\mathrm{LT}^{\mathrm{app}}_\sigma(v_1) \ldots \mathrm{LT}^{\mathrm{app}}_\sigma(v_r)\}$.

(B7) If $\partial\mathcal{O} \not\subseteq U$ then increase d by one, update $U := \langle \mathbb{T}^n_{\leq d} \rangle_K$, and go to (B3).

(B8) Apply the Final Reduction Algorithm and return its result $(g_1, \ldots, g_\nu)$.

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