

Contents

Foreword	V
Introduction	1
0.1 What Is This Book About?	1
0.2 What Is a Gröbner Basis?	2
0.3 Who Invented This Theory?	3
0.4 Now, What Is This Book <i>Really</i> About?	4
0.5 What Is This Book <i>Not</i> About?	7
0.6 Are There any Applications of This Theory?	8
0.7 How Was This Book Written?	10
0.8 What Is a Tutorial?	11
0.9 What Is CoCoA?	12
0.10 And What Is This Book Good for?	12
0.11 Some Final Words of Wisdom	13
1. Foundations	15
1.1 Polynomial Rings	17
<i>Tutorial 1.</i> Polynomial Representation I	24
<i>Tutorial 2.</i> The Extended Euclidean Algorithm	26
<i>Tutorial 3.</i> Finite Fields	27
1.2 Unique Factorization	29
<i>Tutorial 4.</i> Euclidean Domains	35
<i>Tutorial 5.</i> Squarefree Parts of Polynomials	37
<i>Tutorial 6.</i> Berlekamp's Algorithm	38
1.3 Monomial Ideals and Monomial Modules	41
<i>Tutorial 7.</i> Cogenerators	47
<i>Tutorial 8.</i> Basic Operations with Monomial Ideals and Modules	48
1.4 Term Orderings	49
<i>Tutorial 9.</i> Monoid Orderings Represented by Matrices	57
<i>Tutorial 10.</i> Classification of Term Orderings	58
1.5 Leading Terms	59
<i>Tutorial 11.</i> Polynomial Representation II	65
<i>Tutorial 12.</i> Symmetric Polynomials	66
<i>Tutorial 13.</i> Newton Polytopes	67

1.6	The Division Algorithm	69
	<i>Tutorial 14.</i> Implementation of the Division Algorithm	73
	<i>Tutorial 15.</i> Normal Remainders	75
1.7	Gradings	76
	<i>Tutorial 16.</i> Homogeneous Polynomials	83
2.	Gröbner Bases	85
2.1	Special Generation	87
	<i>Tutorial 17.</i> Minimal Polynomials of Algebraic Numbers	89
2.2	Rewrite Rules	91
	<i>Tutorial 18.</i> Algebraic Numbers	97
2.3	Syzygies	99
	<i>Tutorial 19.</i> Syzygies of Elements of Monomial Modules	108
	<i>Tutorial 20.</i> Lifting of Syzygies	108
2.4	Gröbner Bases of Ideals and Modules	110
2.4.A	Existence of Gröbner Bases	111
2.4.B	Normal Forms	113
2.4.C	Reduced Gröbner Bases	115
	<i>Tutorial 21.</i> Linear Algebra	119
	<i>Tutorial 22.</i> Reduced Gröbner Bases	119
2.5	Buchberger's Algorithm	121
	<i>Tutorial 23.</i> Buchberger's Criterion	127
	<i>Tutorial 24.</i> Computing Some Gröbner Bases	129
	<i>Tutorial 25.</i> Some Optimizations of Buchberger's Algorithm	130
2.6	Hilbert's Nullstellensatz	133
2.6.A	The Field-Theoretic Version	134
2.6.B	The Geometric Version	137
	<i>Tutorial 26.</i> Graph Colourings	143
	<i>Tutorial 27.</i> Affine Varieties	143
3.	First Applications	145
3.1	Computation of Syzygy Modules	148
	<i>Tutorial 28.</i> Splines	155
	<i>Tutorial 29.</i> Hilbert's Syzygy Theorem	159
3.2	Elementary Operations on Modules	160
3.2.A	Intersections	162
3.2.B	Colon Ideals and Annihilators	166
3.2.C	Colon Modules	169
	<i>Tutorial 30.</i> Computation of Intersections	174
	<i>Tutorial 31.</i> Computation of Colon Ideals and Colon Modules	175
3.3	Homomorphisms of Modules	177
3.3.A	Kernels, Images, and Liftings of Linear Maps	178
3.3.B	Hom-Modules	181
	<i>Tutorial 32.</i> Computing Kernels and Pullbacks	191
	<i>Tutorial 33.</i> The Depth of a Module	192

3.4	Elimination	195
	<i>Tutorial 34.</i> Elimination of Module Components	202
	<i>Tutorial 35.</i> Projective Spaces and Graßmannians	204
	<i>Tutorial 36.</i> Diophantine Systems and Integer Programming	207
3.5	Localization and Saturation	211
	3.5.A Localization	212
	3.5.B Saturation	215
	<i>Tutorial 37.</i> Computation of Saturations	220
	<i>Tutorial 38.</i> Toric Ideals	221
3.6	Homomorphisms of Algebras	225
	<i>Tutorial 39.</i> Projections	234
	<i>Tutorial 40.</i> Gröbner Bases and Invariant Theory	236
	<i>Tutorial 41.</i> Subalgebras of Function Fields	239
3.7	Systems of Polynomial Equations	241
	3.7.A A Bound for the Number of Solutions	243
	3.7.B Radicals of Zero-Dimensional Ideals	246
	3.7.C Solving Systems Effectively	254
	<i>Tutorial 42.</i> Strange Polynomials	261
	<i>Tutorial 43.</i> Primary Decompositions	263
	<i>Tutorial 44.</i> Modern Portfolio Theory	267
A.	How to Get Started with CoCoA	275
B.	How to Program CoCoA	283
C.	A Potpourri of CoCoA Programs	293
D.	Hints for Selected Exercises	305
	Notation	309
	Bibliography	313
	Index	315