IDEALIZATION OF MODULES IN COMPUTER ALGEBRA

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Abstract. Based on an explicit description of the idealization of a graded submodule of a graded free module, we examine the behaviour of Gröbner bases and minimal homogeneous systems of generators under this process. Then we show how one can idealize a homogeneous presentation. Using this theory, we present a unified description of several strategies for computing minimal homogeneous presentations and minimal graded free resolutions, in particular of the vertical and horizontal strategies. We obtain at a simple and compact algorithm for computing minimal graded free resolutions degree-by-degree which lends itself well to further optimizations.

1. Introduction

One of the central and notoriously difficult tasks in Computational Commutative Algebra is the computation of a minimal graded free resolution of a finitely generated graded module. The implementators of the major computer algebra systems spent considerable energy on this problem and produced a number of different strategies for achieving an efficient solution (see for instance [2], [6], [3], [10] and [12]). Since the efficiency of a particular strategy strongly depends on the nature of the examples to which it is applied, it has proven difficult to determine an optimal strategy. Furthermore, inspired by the actual implementation, the mathematical description of these strategies frequently uses specially adapted terminologies. This fact makes it hard to compare the various approaches.

In this paper we propose a unified framework for treating algorithms which compute minimal homogeneous presentations or minimal graded free resolutions. It is based on the notion of the idealization of a module. In commutative algebra idealizations were first used by M. Nagata in [11]. In other areas of mathematics they are known under different names, e.g. one-point-extensions. The basic idea is to construct a ring which contains a given module as an ideal. In this way computations for the module become computations for that ideal. In our case we follow an idea pioneered in [1] and [6]: we idealize not only the module under consideration but also the graded free modules appearing in its minimal homogeneous presentation or its minimal graded free resolution. Thus the computation of the minimal graded free resolution becomes a Gröbner basis computation in a large polynomial ring. Different strategies for computing resolutions turn out to be nothing but different selection strategies for that Gröbner basis computation.

Besides being able to unify and explain several approaches we will also formulate our own version of the "horizontal" (i.e. degree-by-degree) strategy for computing minimal homogeneous presentations or minimal graded free resolutions. With this algorithm we obtain a simplicity and compactness which is a good basis for efficient implementations.

Now we describe the contents of the paper in more detail. In the second section we recall the construction of the idealization of a module and apply it to the cases of graded free modules and their graded submodules. After finding an explicit description of the ideal representing the idealization of the module, we study the connections between Gröbner bases and minimal systems of generators of the module and its idealization ideal.

In Section 3 we proceeed to idealize the entire minimal homogeneous presentation of the module. The ideal representing this idealization is shown to contain both the minimal homogeneous system of generators of the module and (via an elimination process) the module of its syzygies. Then we use this description in Section 4 to compute minimal homogeneous presentations. We present two strategies for this purpose: the vertical and the horizontal strategy. The vertical strategy proceeds by first computing the minimal homogeneous system of generators of the module and then the minimal homogeneous system of generators of its syzygy module. The horizontal strategy computes both systems simultaneously and proceeds degree-by-degree.

Both the vertical and the horizontal strategy can be generalized to compute the minimal graded free resolution of the module. This is done in the last section where we obtain algorithms which lend themselves well to further study and optimization. To keep the paper uniform and accessible, we have adhered to the definitions and notation in our book [7] and in the foundational paper [9]. Explicit computations of examples for the algorithms introduced here are contained in [8].

2. Idealization of Graded Modules

Let us begin by defining the idealization of a module and collecting some basic properties of this process.

Definition 2.1. Let R be a ring and M an R-module. We equip the product set $R\times M$ with two operations. The addition $+ : (R\times M)\times(R\times M) \longrightarrow R\times M$ is given by $(r,m)+(r',m') = (r+r', m+m')$ and the multiplication $\cdot : (R\times M)\times(R\times M) \longrightarrow R\times M$ is given by $(r,m)\cdot(r',m') = (rr', rm'+r'm)$ for $r, r' \in R$ and $m, m' \in M$. In this way, the set $R \times M$ becomes a commutative ring with identity element $(1,0)$. We call this ring the **idealization** of M and denote it by $R \ltimes M$.

The ring $R \ltimes M$ is an R-algebra via the ring homomorphism $R \longrightarrow R \ltimes M$ given by $r \mapsto (r,0)$. The canonical map $\imath : M \longrightarrow R \ltimes M$ defined by $\imath(m) = (0,m)$ is injective and R-linear. Its image is an ideal in $R \ltimes M$. We shall denote this ideal by $\imath(M)$ and identify the elements of M with their images under $\imath$.

Remark 2.2. Let R be a ring and M an R-module.

a) The ideal $\imath(M)$ satisfies $\imath(M)^2 = 0$.
b) Given an R-submodule $N \subseteq M$, the inclusion $R \times N \subseteq R \times M$ defines an injective ring homomorphism $R \ltimes N \longrightarrow R \ltimes M$. The image of N in $R \ltimes M$ is an ideal which is contained in the ideal $\imath(M)$.
c) Let Γ be a monoid, let R be Γ-graded, and let M be a Γ-graded R-module. Then $R \ltimes M$ is a Γ-graded ring via $(R \ltimes M)_\gamma = R_\gamma \times M_\gamma$ for all $\gamma \in \Gamma$, and $\imath(M)$ is a homogeneous ideal in this ring.

In the sequel we want to study the idealization of a graded submodule of a graded free module. In the remainder of this paper we use the following setting. Let K be a field, and let the polynomial ring $P = K[x_1, \ldots, x_n]$ be positively graded by a matrix $W \in \mathrm{Mat}_{m,n}(\mathbb{Z})$. Thus P is equipped with a $\mathbb{Z}^m$ grading such that $1 \in P_0$, the indeterminate x_i is homogeneous, and the degree of x_i (given by the i^{th} column of W) is lexicographically positive.

Furthermore, let $d_{01}, \ldots, d_{0r} \in \mathbb{Z}^m$, let $F_0 = \oplus_{i=1}^r P(-d_{0i})$, and let M be a graded submodule of F_0. The canonical basis vectors of F_0 will be denoted by $e_1, \ldots, e_r$. By considering $e_1, \ldots, e_r$ as indeterminates, we form the polynomial ring $\overline{P} = K[x_1, \ldots, x_n, e_1, \ldots, e_r] = P[e_1, \ldots, e_r]$. We equip $\overline{P}$ with the grading given by $\overline{W} = (W \mid d_{01} \ \cdots \ d_{0r}) \in \mathrm{Mat}_{m,n+r}(\mathbb{Z})$. A vector $v = f_1e_1 + \cdots + f_re_r \in F_0$ will be identified with the corresponding polynomial in $\overline{P}$.

Moreover, we let $\mathfrak{e}$ be the ideal generated by $E = \{e_ie_j \mid 1 \leq i \leq j \leq r\}$ in $\overline{P}$, and let $\mathcal{E}$ be the tuple

$$\mathcal{E} = (e_1e_1, e_1e_2, \ldots, e_1e_r, e_2e_2, e_2e_3, \ldots, e_2e_r, \ldots, e_re_r) \in \overline{P}^{\,r(r+1)/2}$$

In this situation we can represent the idealization of M as follows.

Proposition 2.3. (Idealization of a Graded Submodule)
Let $\mathcal{V} = (v_1, \ldots, v_s)$ be a tuple of homogeneous vectors which generate M.

a) The map $\varphi : P \ltimes F_0 \longrightarrow \overline{P}/\mathfrak{e}$ which sends $(f, (g_1, \ldots, g_r))$ to the residue class of $f + g_1e_1 + \cdots + g_re_r$ is an isomorphism of graded rings.

b) Under the composition $M \xrightarrow{\iota} P \ltimes M \hookrightarrow P \ltimes F_0 \xrightarrow{\varphi} \overline{P}/\mathfrak{e}$, the module M is identified with the residue class ideal of $I_M = (v_1, \ldots, v_s) + \mathfrak{e} \subseteq \overline{P}$.

Proof. First we prove a). Since φ is obviously R-linear, we check that φ is compatible with multiplication. Let $(f, (g_1, \ldots, g_r))$ and $(f', (g'_1, \ldots, g'_r))$ be two elements of $R \ltimes M$. Computing modulo $\mathfrak{e}$, we get

$$\begin{aligned}
\varphi((f, (g_1, \ldots, g_r)) \cdot (f', (g'_1, \ldots, g'_r))) &= \varphi((ff', (fg'_1 + f'g_1, \ldots, fg'_r + f'g_r))) \\
&= ff' + (fg'_1 + f'g_1)e_1 + \cdots + (fg'_r + f'g_r)e_r \\
&= (f + g_1e_1 + \cdots + g_re_r) \cdot (f' + g'_1e_1 + \cdots + g'_re_r) \\
&= \varphi((f, (g_1, \ldots, g_r))) \cdot \varphi((f', (g'_1, \ldots, g'_r)))
\end{aligned}$$

Since φ is clearly both injective and surjective, it remains to show that φ is homogeneous. The monomial ideal $\mathfrak{e}$ is homogeneous. Given a homogeneous element $(f, (g_1, \ldots, g_r))$ of $R \ltimes M$, the residue class of $f + g_1e_1 + \cdots + g_re_r$ is homogeneous of the same degree. Therefore φ is a homomorphism of graded rings.

To prove b), we combine the descriptions of the maps in Definition 2.1.b, Remark 2.2.b, and part a). For $i = 1, \ldots, s$, we see that v_i is mapped to the residue class $v_i + \mathfrak{e}$ by the composition. Hence the claim follows. □

Definition 2.4. In the above setting, the ideal $I_M = (v_1, \ldots, v_s) + \mathfrak{e}$ of $\overline{P}$ is called the **idealization ideal**, or simply the **ideal** of M.

Gröbner bases of graded submodules and Gröbner bases of their ideals are related as follows. A module term ordering σ on $\mathbb{T}^n\langle e_1, \ldots, e_r\rangle$ is called **compatible** with a term ordering τ on $\mathbb{T}^n$ if $t \geq_\tau t'$ implies $te_i \geq_\sigma t'e_i$ for all $t, t' \in \mathbb{T}^n$ and $i \in \{1, \ldots, r\}$. We shall say that a term ordering $\overline{\sigma}$ on $\mathbb{T}(x_1, \ldots, x_n, e_1, \ldots, e_r)$ **extends** both σ and τ if the restriction of $\overline{\sigma}$ to $\mathbb{T}^n$ is τ and its restriction to $\mathbb{T}^n\langle e_1, \ldots, e_r\rangle$ is σ.

Proposition 2.5. (Gröbner Bases and Idealization)
Let $I_M \subseteq \overline{P}$ be the ideal of M. Furthermore, let τ be a term ordering on $\mathbb{T}^n$, let σ be a module term ordering on $\mathbb{T}^n\langle e_1, \ldots, e_r\rangle$ which is compatible with τ, and let $\overline{\sigma}$ be a term ordering on $\mathbb{T}(x_1, \ldots, x_n, e_1, \ldots, e_r)$ which extends both σ and τ.

a) Let G be a σ-Gröbner basis of M. Then $G \cup E$ is a $\overline{\sigma}$-Gröbner basis of I_M.
b) Let G be the reduced σ-Gröbner basis of M. Then the reduced $\overline{\sigma}$-Gröbner basis of I_M is $G \cup \{e_i e_j \in E \mid e_i, e_j \notin \mathrm{LT}_\sigma(M)\}$.

Proof. To prove a), we let $G = \{g_1, \ldots, g_{s'}\}$ and $f \in I_M$. By subtracting suitable multiples of polynomials $e_i e_j$, we see that we may reduce f to the form $h = h_0 + h_1 e_1 + \cdots + h_r e_r$ with $h_0, \ldots, h_r \in P$. Since I_M is contained in $(e_1, \ldots, e_r)$, we have $h_0 = 0$. The polynomial $h = h_1 e_1 + \cdots + h_r e_r \in I$ is linear in $e_1, \ldots, e_r$. The only generators of I_M which are linear in $e_1, \ldots, e_r$ are $v_1, \ldots, v_s$. Hence h is a P-linear combination of those generators. Thus we have $(h_1, \ldots, h_r) \in \langle v_1, \ldots, v_s\rangle$ and $\mathrm{LT}_\sigma((h_1, \ldots, h_r)) = t\ \mathrm{LT}_\sigma(g_i)$ for some $t \in \mathbb{T}^n$ and $i \in \{1, \ldots, s'\}$. Since the term ordering $\overline{\sigma}$ extends σ, we get $\mathrm{LT}_{\overline{\sigma}}(h) = t\ \mathrm{LT}_{\overline{\sigma}}(g_i)$. Therefore we can reduce h by G. After several such reduction steps we eventually get zero, because G is a σ-Gröbner basis of M. Altogether, the polynomial f reduces to zero using $G \cup E$.

To show claim b), we let $\overline{G}$ be the reduced $\overline{\sigma}$-Gröbner basis of I_M. Since we have $e_i e_j \in I_M$, we know $e_i e_j \in \mathrm{LT}_{\overline{\sigma}}(I_M)$ for all $1 \leq i \leq j \leq r$. Now it follows from the additional condition $e_i, e_j \notin \mathrm{LT}_\sigma(M)$ that $e_i e_j$ is a minimal generator of $\mathrm{LT}_{\overline{\sigma}}(I_M)$, and therefore $e_i e_j$ is contained in $\overline{G}$. Thus $\overline{G}$ has the form $\overline{G} = G \cup \{e_i e_i \in E \mid e_i, e_j \notin \mathrm{LT}_\sigma(I)\}$ for some set of polynomials $G \subseteq \overline{P}$. The fact that the polynomials in G are irreducible with respect to E implies that they have degree ≤ 1 in the indeterminates $e_1, \ldots, e_r$. Since we have $I_M \subseteq (e_1, \ldots, e_r)$, this shows that the polynomials in G are linear forms in $e_1, \ldots, e_r$. Since the Buchberger Algorithm preserves homogeneity, they are homogeneous with respect to the grading given by $\overline{W}$. Hence they are actually images of homogeneous vectors in M. Their leading terms generate all elements in $\mathrm{LT}_\sigma(I_M)$ of degree one in $e_1, \ldots, e_r$. Thus the leading terms of the corresponding elements of M generate $\mathrm{LT}_\sigma(M)$. Finally, since the elements of G are fully reduced against each other, the corresponding elements of M are fully reduced, too. Hence G is the image of the reduced σ-Gröbner basis of M in $\overline{P}$. □

In particular, notice that the reduced $\overline{\sigma}$-Gröbner basis of I_M is $G \cup E$ if we have $M \subseteq (x_1, \ldots, x_n)\, F_0$ and G is the reduced σ-Gröbner basis of M. Let us give some typical examples of simultaneous extensions $\overline{\sigma}$ of σ and τ as required by this proposition (for a classification, see [5]).

Remark 2.6. Let $\tau = \mathrm{Ord}(V)$ be a term ordering on $\mathbb{T}^n$ given by a non-singular matrix $V \in \mathrm{Mat}_n(\mathbb{Z})$.

a) Let σ be the module term ordering $\tau - \mathtt{Pos}$ on $\mathbb{T}^n\langle e_1, \ldots, e_r\rangle$ (see [7], 1.4.16). Then $\overline{\sigma} = \mathrm{Ord}\left(\begin{smallmatrix} V & 0 \\ 0 & \mathcal{I}_r \end{smallmatrix}\right)$ is a term ordering on $\mathbb{T}^n(x_1, \ldots, x_n, e_1, \ldots, e_r)$ which extends both τ and σ.
b) Let σ be the module term ordering $\mathtt{Pos}{-}\tau$ on $\mathbb{T}^n\langle e_1, \ldots, e_r\rangle$. Then $\overline{\sigma} = \mathrm{Ord}\left(\begin{smallmatrix} 0 & \mathcal{I}_r \\ V & 0 \end{smallmatrix}\right)$ is a term ordering on $\mathbb{T}^n(x_1, \ldots, x_n, e_1, \ldots, e_r)$ which extends both τ and σ.
c) Suppose that τ is a degree compatible term ordering of the form $\tau = \mathrm{Ord}\left(\begin{smallmatrix} W \\ W' \end{smallmatrix}\right)$ with $W' \in \mathrm{Mat}_{n-m,n}(\mathbb{Z})$ and that $d_{0i} >_{\mathtt{Lex}} 0$ for $i = 1, \ldots, r$. Then the module

term ordering $\sigma = \texttt{Deg}{-}\tau{-}\texttt{Pos}$ is compatible with τ, and the term ordering $\overline{\sigma} = \begin{pmatrix} W & d_{01} \cdots d_{0r} \\ W' & 0 \cdots 0 \\ 0 & \mathcal{I}_r \end{pmatrix}$ on $\mathbb{T}(x_1, \ldots, x_n, e_1, \ldots, e_r)$ extends both τ and σ.

Example 2.7. Let $P = K[x, y, z]$ be standard graded, and let τ be the term ordering `DegRevLex` on $\mathbb{T}^3$. Then $\sigma = \texttt{PosDegRevLex}$ is a module term ordering on $\mathbb{T}^3\langle e_1, e_2, e_3\rangle$ which is compatible with τ. Moreover, the term ordering $\overline{\sigma} = \mathrm{Ord}\left(\begin{smallmatrix} 0 & \mathcal{I}_3 \\ V & 0 \end{smallmatrix}\right)$ with $V = \left(\begin{smallmatrix} 1 & 1 & 1 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{smallmatrix}\right)$ extends both σ and τ.

Now let M be the graded submodule of $P(-1)^3$ generated by the set of vectors $\{(x, y, 0), (0, 1, 0), (0, z^2, xy)\}$. Then the reduced σ-Gröbner basis of M is $\{e_2, xe_1, xye_3\}$. By applying the proposition, we see that the reduced $\overline{\sigma}$-Gröbner basis of the ideal I_M is $\{e_2, xe_1, xye_3, e_1^2, e_1e_3, e_3^2\}$.

After clarifying the behaviour of Gröbner bases with respect to idealization, we turn to minimal homogeneous systems of generators.

Proposition 2.8. (Minimal Generators and Idealization)
Let M be a graded submodule of a graded free P-module $F_0 = \oplus_{i=1}^r P(-d_{0i})$ such that $M \subseteq (x_1, \ldots, x_n)\, F_0$ and where $d_{0i} >_{\texttt{Lex}} 0$ for $i = 1, \ldots, r$ Furthermore, let $\mathcal{V} = (v_1, \ldots, v_s)$ be a tuple of non-zero homogeneous vectors which generate M.

a) Assume that $\mathcal{V}$ is a minimal homogeneous system of generators of M. Then the set $\{v_1, \ldots, v_s\} \cup E$ is a minimal homogeneous system of generators of I_M.

Let τ be a term ordering on $\mathbb{T}^n$, let σ be a module term ordering on $\mathbb{T}^n\langle e_1, \ldots, e_r\rangle$ which is compatible with τ, and let $\overline{\sigma}$ be a term ordering on $\mathbb{T}(x_1, \ldots, x_n, e_1, \ldots, e_r)$ which extends both σ and τ.

b) If we apply Buchberger's Algorithm with Minimalization (see [8]*, 4.6.3 or* [9]*, Cor. 13) to the tuple $(\mathcal{V} \mid \mathcal{E})$, it computes a minimal system of generators of the ideal $I_M \subseteq \overline{P}$ of the form $(\mathcal{V}_{\min} \mid \mathcal{E})$, where $\mathcal{V}_{\min}$ is a minimal homogeneous system of generators of M.*

c) If we apply the variant of Buchberger's Algorithm with Minimalization (see [8]*, 4.6.5 or* [9]*, Rem. 14.b) to the tuple $(\mathcal{V} \mid \mathcal{E})$, it computes a homogeneous $\overline{\sigma}$-Gröbner basis of I_M of the form $(\mathcal{G} \mid \mathcal{E})$ and a minimal system of generators of the form $(\mathcal{G}_{\min} \mid \mathcal{E})$, where $\mathcal{G}$ is a σ-Gröbner basis of M and $\mathcal{G}_{\min}$ is a minimal homogeneous system of generators of M which is contained in $\mathcal{G}$.*

Proof. First we show a). By definition, the set $\{v_1, \ldots, v_s\} \cup E$ generates I_M. Consider the grading on $\overline{P}$ defined by $\deg(x_i) = 0$ and $\deg(e_j) = 1$ for $i = 1, \ldots, n$ and $j = 1, \ldots, r$. Then I_M is a homogeneous ideal with respect to this grading. Its homogeneous component of degree one is $Pv_1 + \cdots + Pv_s$, so that it follows from the minimality of $\mathcal{V}$ that we cannot drop any of these generators in $(\mathcal{V} \mid \mathcal{E})$. Next we consider the image of I_M under the substitution $x_i \mapsto 0$ for $i = 1, \ldots, n$. Since $M \subseteq (x_1, \ldots, x_n)\, F_0$, we have $v_i \mapsto 0$ for $i = 1, \ldots, s$. The images of the elements e_ie_j are minimal generators. Therefore the elements e_ie_j are minimal generators of I_M.

Now we prove b). The hypothesis $d_{0i} >_{\texttt{Lex}} 0$ implies that $\overline{P}$ is positively graded. Hence we can apply Buchberger's Algorithm with Minimalization. Since $(\mathcal{V} \mid \mathcal{E})$ generates I_M, the algorithm computes a minimal system of generators of I_M which is contained in $(\mathcal{V} \mid \mathcal{E})$. By considering the grading $\deg(x_i) = 0$, $\deg(e_j) = 1$ again,

we see that the vectors in $\mathcal{V}$ chosen by the algorithm correspond to a minimal system of generators of M. Moreover, the algorithm has to choose all polynomials e_ie_j because they are all minimal generators of I_M.

To show c), we claim that the $\overline{\sigma}$-Gröbner basis of I_M computed by the algorithm has the form $(\mathcal{G} \mid \mathcal{E})$. Clearly, a critical pair (i,j) such that $\mathrm{LT}_{\overline{\sigma}}(v_i) = c_it_ie_{\gamma_i}$ and $\mathrm{LT}_{\overline{\sigma}}(v_j) = c_jt_je_{\gamma_j}$ satisfy $\gamma_i = \gamma_j$ corresponds to a critical pair of the vectors $v_1, \ldots, v_s$ in M. However, if we have $\gamma_i \neq \gamma_j$, then the corresponding S-vector has degree two in the indeterminates $e_1, \ldots, e_r$. It reduces to zero, because the terms in its support are of the form te_ke_ℓ where e_ke_ℓ has a smaller degree and has been appended to the Gröbner basis already. Moreover, every time the algorithm encounters a term e_ke_ℓ, that term is irreducible with respect to the part of the Gröbner basis computed so far and is appended to the Gröbner basis and to the minimal system of generators. Altogether, we see that the algorithm computes a Gröbner basis of the form $(\mathcal{G} \mid \mathcal{E})$. As above, it follows that the minimal set of generators it finds is of the form $(\mathcal{G}_{\min} \mid \mathcal{E})$. □

Let us briefly discuss the hypotheses of this proposition.

Remark 2.9. Suppose we are in the setting of the proposition.

a) The assumption $M \subseteq (x_1, \ldots, x_n)\, F_0$ is essential for the proposition to be true. Otherwise, S-vectors of the form $e_{\gamma_i}v_j - e_{\gamma_j}v_i$ could be appended to the Gröbner basis, because they contain terms of the form $c\, e_ke_\ell$ with $c \in K$ which have not yet been treated, i.e. which are not yet in the Gröbner basis.
b) The assumption $d_{0i} >_{\mathtt{Lex}} 0$ for $i = 1, \ldots, r$ is used to conclude that the grading on $\overline{P}$ given by $\overline{W}$ is positive, but it is not essential for the correctness of the algorithm (see also the discussion below).

3. Idealization of Homogeneous Presentations

In this section we want to idealize not only a graded submodule, but a whole homogeneous presentation. We continue to assume that M is a graded submodule of a graded free P-module $F_0 = \oplus_{i=1}^r P(-d_{0i})$, where $d_{01}, \ldots, d_{0r} \in \mathbb{Z}^m$, and that $\mathcal{V} = (v_1, \ldots, v_s)$ is a tuple of non-zero homogeneous elements which generate M. Moreover, it will be convenient to assume that $\mathcal{V}$ is **deg-ordered**, i.e. that we have $\deg_W(v_1) \leq_{\mathtt{Lex}} \cdots \leq_{\mathtt{Lex}} \deg_W(v_s)$.

Then we form the graded free P-module $F_1 = \oplus_{j=1}^s P(-d_{1j})$ where $d_{1j} = \deg_W(v_j)$ for $j = 1, \ldots, s$. We let $\{\varepsilon_1, \ldots, \varepsilon_s\}$ be the canonical basis of F_1 and consider a homogeneous presentation

$$F_2 \xrightarrow{\psi} F_1 \xrightarrow{\varphi} M \longrightarrow 0$$

of M where $\varphi(\varepsilon_j) = v_j$ for $j = 1, \ldots, s$ and where F_2 is a another graded free P-module.

Remark 3.1. For every $d \in \mathbb{Z}^m$, a homogeneous presentation of the shifted module $M(d)$ is given by

$$F_2(d) \xrightarrow{\tilde{\psi}} F_1(d) \xrightarrow{\tilde{\varphi}} M(d) \longrightarrow 0$$

where $F_1(d)$ and $F_2(d)$ are graded free P-modules, and where the homogeneous P-linear maps $\tilde{\varphi}$ and $\tilde{\psi}$ are given by the same homogeneous matrices as φ and ψ, respectively. Therefore the computation of a minimal homogeneous presentation

of M is equivalent to the computation of a minimal homogeneous presentation of $M(d)$.

In view of this remark, we shall from now on assume that $d_{0i} >_{\mathtt{Lex}} 0$ for $i = 1, \ldots, r$. Notice that this implies $d_{1j} >_{\mathtt{Lex}} 0$ for $j = 1, \ldots, s$. This assumption will prove useful in our theoretical discussion below. In an actual implementation it is not really necessary, because shifting the entire computation by a fixed degree does not change its correctness.

In order to idealize the above presentation of M, we introduce the polynomial ring $\widetilde{P} = P[e_1, \ldots, e_r, \varepsilon_1, \ldots, \varepsilon_s] = K[x_1, \ldots, x_n, e_1, \ldots, e_r, \varepsilon_1, \ldots, \varepsilon_s]$ and equip it with the grading given by $\widetilde{W} = (W \mid d_{01} \cdots d_{0r} \mid d_{11} \cdots d_{1s})$. Notice that, because of our assumption, this grading is positive. By Proposition 2.3.a, the idealization of the module $F_0 \oplus F_1$ is the ring $\widetilde{P}/\tilde{\mathfrak{e}}$, where $\tilde{\mathfrak{e}}$ is the homogeneous ideal generated by the union of $\{e_i e_j \mid i, j = 1, \ldots, r\}$, $\{e_i \varepsilon_j \mid i \in \{1, \ldots, r\}, j \in \{1, \ldots, s\}\}$, and $\{\varepsilon_i \varepsilon_j \mid i, j = 1, \ldots, s\}$.

Let $\imath_0 : F_0 \longrightarrow \widetilde{P}/\tilde{\mathfrak{e}}$ be the homogeneous injective P-linear map given by $\imath_0((f_1, \ldots, f_r)) = f_1 e_1 + \cdots f_r e_r + \tilde{\mathfrak{e}}$, and let $\imath_1 : F_1 \longrightarrow \widetilde{P}/\tilde{\mathfrak{e}}$ be the homogeneous injective P-linear map given by $\imath_1((f_1, \ldots, f_s)) = f_1 \varepsilon_1 + \cdots f_s \varepsilon_s + \tilde{\mathfrak{e}}$. By [7], 3.6.1, if $(f_1, \ldots, f_s) \in \mathrm{Syz}_P(\mathcal{V})$ is a homogeneous syzygy of $\mathcal{V}$, then the corresponding element $f_1 \varepsilon_1 + \cdots + f_s \varepsilon_s$ of $\widetilde{P}$ is contained in the ideal $(v_1 - \varepsilon_1, \ldots, v_s - \varepsilon_s)$.

Definition 3.2. The ideal $\tilde{I}_M = (v_1 - \varepsilon_1, \ldots, v_s - \varepsilon_s) + \tilde{\mathfrak{e}}$ in $\widetilde{P}$ is called the **ideal of the presentation** of M given above.

Our next proposition serves to justify this name.

Proposition 3.3. (Idealization of a Homogeneous Presentation)
Let $\tilde{I}_M \subseteq \widetilde{P}$ be the ideal of the presentation of M.

a) There exists a unique P-algebra homomorphism

$$\Phi : P[\varepsilon_1, \ldots, \varepsilon_s]/(\varepsilon_i \varepsilon_j)_{i,j=1,\ldots,s} \longrightarrow \overline{P}/\mathfrak{e}$$

which maps the residue class of ε_i to $v_i + \mathfrak{e}$ for $i = 1, \ldots, s$.

b) The image of Φ is the residue class ideal of the ideal of M.

c) The kernel of Φ is the residue class ideal of the ideal of $\mathrm{Syz}_P(\mathcal{V})$.

d) The ideal of $\mathrm{Syz}_P(\mathcal{V})$ *is given by $\tilde{I}_M \cap P[\varepsilon_1, \ldots, \varepsilon_s]$.*

Proof. To prove a), it suffices to note that Φ is well-defined, because we have $v_i v_j \in \mathfrak{e}$ for $i, j = 1, \ldots, s$. Claim b) follows from Proposition 2.3.b.

To show c), we note that Φ is induced by the substitution homomorphism $\Psi : P[\varepsilon_1, \ldots, \varepsilon_s] \longrightarrow \overline{P}/\mathfrak{e}$ with $\varepsilon_k \mapsto v_k + \mathfrak{e}$ for $k = 1, \ldots, s$. The ideal $(\varepsilon_k \varepsilon_\ell)_{k,\ell=1,\ldots,s}$ is contained in $\ker(\Psi)$. Since $v_1, \ldots, v_s \in (e_1, \ldots, e_r)$, the kernel of the composition of Ψ with the canonical surjection $\overline{P}/\mathfrak{e} \longrightarrow P$ is $(\varepsilon_1, \ldots, \varepsilon_s)$, and therefore we have $\ker(\Psi) \subseteq (\varepsilon_1, \ldots, \varepsilon_s)$. Furthermore, we have $f_1 \varepsilon_1 + \cdots + f_s \varepsilon_s \in \ker(\Psi)$ if and only if $f_1 v_1 + \cdots + f_s v_s = 0$. Hence $\{f_1 \varepsilon_1 + \cdots + f_s \varepsilon_s \mid (f_1, \ldots, f_s) \in \mathrm{Syz}_P(\mathcal{V})\} \cup \{\varepsilon_k \varepsilon_\ell \mid k, \ell = 1, \ldots, s\}$ generates $\ker(\Psi)$, and the claim follows from Proposition 2.3.b.

Finally, part d) follows from c) by a slight generalization of [7], 3.6.2, but the proof of that proposition still applies without modifications. □

4. Computation of Minimal Homogeneous Presentations

As above, we assume that M is a submodule of a graded free module $F_0 = \oplus_{i=1}^r P(-d_{0i})$ and that it is given by a deg-ordered homogeneous system of generators $\mathcal{V} = (v_1, \ldots, v_s)$. We want to compute a minimal homogeneous presentation of M. The graded version of Nakayama's lemma implies that we may assume $M \subseteq (x_1, \ldots, x_n)\, F_0$. There is an obvious method to solve our task.

Remark 4.1. Consider the following instructions.

1) Using the Buchberger Algorithm with Minimalization (see [8], 4.6.3), compute a subtuple $\mathcal{V}_{\min}$ of $\mathcal{V}$ of homogeneous vectors which generate M minimally.
2) Using the lifting of syzygies (see [7], 3.1.8), compute a matrix $\mathcal{S}$ whose columns generate the module $\mathrm{Syz}_P(\mathcal{V}_{\min})$. (Notice that all computations involved here keep the homogeneity of the input. Thus $\mathcal{S}$ is a homogeneous matrix.)
3) Using the Buchberger Algorithm with Minimalization, compute a subtuple $\mathcal{S}_{\min}$ of $\mathcal{S}$ of homogeneous vectors which generate $\mathrm{Syz}_P(\mathcal{S}_{\min})$ minimally.
4) Return $\mathcal{V}_{\min}$ and $\mathcal{S}_{\min}$ and stop.

This is an algorithm which computes two tuples $\mathcal{V}_{\min}$ and $\mathcal{S}_{\min}$ of homogeneous vectors such that $\mathcal{V}_{\min}$ generates M minimally and $\mathcal{S}_{\min}$ generates $\mathrm{Syz}_P(\mathcal{V}_{\min})$ minimally. In particular, we have a minimal homogeneous presentation

$$F_2 \xrightarrow{\psi} F_1 \xrightarrow{\varphi} M \longrightarrow 0$$

where F_1 and F_2 are graded free P-modules, where ψ is given by $\mathcal{S}_{\min}$ and where φ is given by $\mathcal{V}_{\min}$.

This algorithm is rather wasteful, because the reduction steps involved in computing the Gröbner basis of M in step 1) have to be done again in step 2). Using the idealization of the presentation discussed in the previous subsection, we can proceed in a more efficient way. The method we use in the next theorem is called the **vertical strategy**. The reason is that we compute first a minimal system of generators $\mathcal{G}_{\min}$ of M by looping through all necessary degrees (which we think of as being on top of each other) and then a minimal system of generators of $\mathrm{Syz}_P(\mathcal{G}_{\min})$ by a similar loop.

To ease the notation, we shall resort to the following convention. Given a tuple $\mathcal{G} = (g_1, \ldots, g_{s'})$, we write $\mathrm{LT}_\sigma(g_i) = t_i\, e_{\gamma_i}$ with $t_i \in \mathbb{T}^n$ and $\gamma_i \in \{1, \ldots, r\}$ for $i = 1, \ldots, s'$. Moreover, we identify the elements of the graded free P-module $F_0 = \oplus_{i=1}^r P(-d_{0i})$ with their canonical images in the polynomial ring $\overline{P} = P[e_1, \ldots, e_r]$ and equip $\overline{P}$ with the grading given by $\overline{W} = (W \mid d_{01} \cdots d_{0r})$.

Theorem 4.2. (Computing Minimal Presentations Vertically)
Let M be a graded submodule of a graded free P-module $F_0 = \oplus_{i=1}^r P(-d_{0i})$ where $d_{0i} >_{\mathtt{Lex}} 0$ for $i = 1, \ldots, r$. Let $\mathcal{V} = (v_1, \ldots, v_s)$ be a deg-ordered tuple of non-zero homogeneous vectors which generate M. Consider the following instructions.

1) Choose a term ordering σ on $\mathbb{T}^n(e_1, \ldots, e_r)$. Let $B = \emptyset$, $\mathcal{W} = \mathcal{V}$, $\mathcal{G} = \emptyset$, $s' = 0$, $\mathcal{G}_{\min} = \emptyset$, $\mu = 0$, and $\mathcal{S} = \emptyset$.
2) Let d be the smallest degree with respect to Lex *of an element in B or in $\mathcal{W}$. Form the subset B_d and the subtuple $\mathcal{W}_d$, and delete their entries from B and $\mathcal{W}$, respectively.*
3) If $B_d = \emptyset$, continue with step 6). Otherwise, chose a pair $(i, j) \in B_d$ and remove it from B_d.

4) Compute the S-vector S_{ij} of g_i and g_j. Then compute $S'_{ij} = \mathrm{NR}_{\sigma,\mathcal{G}}(S_{ij})$. If $S'_{ij} = 0$ continue with step 3). If $S'_{ij} \neq 0$ and it does not involve the indeterminates $e_1, \ldots, e_r$, append it to $\mathcal{S}$ and continue with step 3).

5) Increase s' by one, append $g_{s'} = S'_{ij}$ to the tuple $\mathcal{G}$, and append the set $\{(i, s') \mid 1 \leq i < s',\ \gamma_i = \gamma_{s'}\}$ to B. Then continue with step 3).

6) If $\mathcal{W}_d = \emptyset$, continue with step 9). Otherwise, choose a vector $v \in \mathcal{W}_d$ and remove it from $\mathcal{W}_d$.

7) Compute $v' = \mathrm{NR}_{\sigma,\mathcal{G}}(v)$ and $\bar{v} = v'(x_1, \ldots, x_n, e_1, \ldots, e_r, 0, \ldots, 0)$. If $\bar{v} = 0$, continue with step 6).

8) Increase s' and μ by one. Adjoin a new indeterminate ε_μ to $\overline{P}$ and extend the grading to this new ring by defining $\deg_{\overline{W}}(\varepsilon_\mu) = \deg_{\overline{W}}(\bar{v})$. Extend the term ordering σ to the new ring in such a way that the extension is an elimination ordering for $\{e_1, \ldots, e_r\}$. Append $g_{s'} = \bar{v} - \varepsilon_\mu$ to $\mathcal{G}$ and $\bar{v}$ to $\mathcal{G}_{\min}$. Append the set $\{(i, s') \mid 1 \leq i < s',\ \gamma_i = \gamma_{s'}\}$ to B. Continue with step 6).

9) If $B \neq \emptyset$ or $\mathcal{W} \neq \emptyset$, continue with step 2).

10) Apply the Buchberger Algorithm with Minimalization to the module generated by $\mathcal{S}$ and obtain a subtuple $\mathcal{S}_{\min}$ of $\mathcal{S}$ which minimally generates that module. Return the pair $(\mathcal{G}_{\min}, \mathcal{S}_{\min})$ and stop.

This is an algorithm which computes a pair $(\mathcal{G}_{\min}, \mathcal{S}_{\min})$ where $\mathcal{G}_{\min}$ is a deg-ordered tuple of homogeneous vectors in F_0 which generate M minimally, and $\mathcal{S}_{\min}$ is a deg-ordered tuple of homogeneous vectors in $\oplus_{i=1}^{\mu} P(-d_{1i})$ which generate $\mathrm{Syz}_P(\mathcal{G}_{\min})$ minimally. Here μ is the number of elements in $\mathcal{G}_{\min}$ and d_{1i} is the degree of the i^{th} element in $\mathcal{G}_{\min}$ for $i = 1, \ldots, \mu$.

Proof. To prove the finiteness of this procedure, we note that elements are appended to $\mathcal{G}$ and B only in steps 5) and 8). By induction on s', we see that all elements of $\mathcal{G}$ are of the form $a_1e_1 + \cdots + a_re_r + b_1\varepsilon_1 + \cdots + b_\mu\varepsilon_\mu$ with $a_i, b_j \in P$. Moreover, the checks in step 4) and step 7) make sure that at least one polynomial a_i is non-zero. Since B is enlarged only when an element is appended to $\mathcal{G}$ which has a new leading term, and since this can happen only finitely many times, the procedure terminates after finitely many steps.

Now we prove correctness. If we substitute $\varepsilon_1 \mapsto 0, \ldots, \varepsilon_\mu \mapsto 0$ everywhere, the algorithm reduces to the Buchberger Algorithm with Minimalization. Thus the resulting tuple $\mathcal{G}_{\min}$ is a minimal homogeneous set of generators of M. To show that $\mathcal{S}_{\min}$ is correct, we use Proposition 3.3. Notice that the elements $g_{s'} = \bar{v} - \varepsilon_\mu$ appended to $\mathcal{G}$ in step 8) are exactly the generators of $\tilde{I}_M$ corresponding to the minimal system of generators $\mathcal{G}_{\min}$ of M. Thus Proposition 3.3.e says that $\mathrm{Syz}_P(\mathcal{G}_{\min})$ corresponds to the residue class ideal of $\tilde{I}_M \cap P[\varepsilon_1, \ldots, \varepsilon_\mu]$ in $P[\varepsilon_1, \ldots, \varepsilon_\mu]/(\varepsilon_1\varepsilon_j \mid i, j = 1, \ldots, \mu)$. Since all extensions of σ are elimination orderings for $\{e_1, \ldots, e_r\}$, the σ-Gröbner basis of $\tilde{I}_M$ contains a system of generators of $\mathrm{Syz}_P(\mathcal{G}_{\min})$.

In fact, we claim that the elements of $\mathcal{S}$ are a σ-Gröbner basis of $\mathrm{Syz}_P(\mathcal{G}_{\min})$ when the algorithm stops. This follows by inspecting the effect of step 4): a σ-Gröbner basis element S'_{ij} of $\tilde{I}_M$ corresponds to a syzygy of $\mathcal{G}_{\min}$ if and only if it is contained in $P[\varepsilon_1, \ldots, \varepsilon_\mu]$. Finally, we note that $\mathcal{S}_{\min}$ is indeed a minimal system of generators of $\mathrm{Syz}_P(\mathcal{G}_{\min})$ because we apply Buchberger's Algorithm with Minimalization in step 10). □

Unlike Remark 4.1, the algorithm described in this theorem returns a minimal system of generators which is not necessarily contained in the given set of generators. Our next remark shows how we can modify the theorem to get this property as well.

Remark 4.3. In the algorithm of the theorem, the elements v of the tuple $\mathcal{V}$ for which the corresponding v' does not involve the indeterminates $e_1, \ldots, e_r$ form a subtuple $\mathcal{V}_{\min}$ of $\mathcal{V}$ which minimally generates M. By keeping track of the representations of new Gröbner basis elements in terms of the elements in $\mathcal{V}_{\min}$, we compute a homogeneous matrix $\mathcal{A}$ such that $\mathcal{G} = \mathcal{V}_{\min}\,\mathcal{A}$. Since $\mathcal{G}_{\min}$ is a subtuple of $\mathcal{G}$, the columns corresponding to the elements of $\mathcal{G}_{\min}$ in this matrix equality yield a matrix $\mathcal{B}$ such that $\mathcal{G}_{\min} = \mathcal{V}_{\min}\,\mathcal{B}$. Then the exact sequence $F_2 \xrightarrow{\psi'} F_1 \xrightarrow{\varphi'} M \longrightarrow 0$, where ψ' is given by the matrix $\mathcal{B}\,\mathcal{S}_{\min}$ and where φ' is given by $\mathcal{V}_{\min}$, is a minimal homogeneous presentation of M.

Sometimes the vertical strategy is not the most efficient way to compute a minimal homogeneous presentation. For instance, suppose that we have some information about the highest degrees occurring in the minimal presentation of M. In this case, it is desirable to compute the minimal generators and their minimal syzygies degree by degree, and to truncate the computation at the appropriate degree. A method which proceeds degree by degree and determines both the minimal generators and their minimal syzygies simultaneously for each degree is called a **horizontal strategy**. Our next theorem shows how we can implement it effectively.

To ease the notation, we shall again resort to the convention $\mathrm{LT}_\sigma(g_i) = t_i e_{\gamma_i}$ with $t_i \in \mathbb{T}^n$ and $\gamma_i \in \{1, \ldots, r\}$. Moreover, we let $\mathrm{LT}_\sigma(h_j) = \tilde{t}_j e_{\eta_j}$ with $\tilde{t}_j \in \mathbb{T}^n$ and $\eta_j \in \{1, \ldots, \mu\}$.

Theorem 4.4. (Computing Minimal Presentations Horizontally)
Let M be a graded submodule of a graded free P-module $F_0 = \oplus_{i=1}^r P(-d_{0i})$, where $d_{0i} >_{\mathtt{Lex}} 0$ for $i = 1, \ldots, r$, and let $\mathcal{V} = (v_1, \ldots, v_s)$ be a deg-ordered tuple of non-zero homogeneous vectors which generate M. Consider the following instructions.

1) *Choose a term ordering σ on the monoid $\mathbb{T}^n(e_1, \ldots, e_r)$. Let $B = \emptyset$, $B' = \emptyset$, $\mathcal{W} = \mathcal{V}$, $\mathcal{G} = \emptyset$, $s' = 0$, $\mathcal{G}_{\min} = \emptyset$, $\mu = 0$, $\mathcal{S} = \emptyset$, $s'' = 0$, and $\mathcal{S}_{\min} = \emptyset$.*
2) *Let d be the smallest degree with respect to* Lex *of an element in $B \cup B'$ or in $\mathcal{W}$. Form the subset B_d of B, the subset B'_d of B', the subtuple $\mathcal{W}_d$ of $\mathcal{W}$, and delete their entries from B, B', and $\mathcal{W}$, respectively.*
3) *If $B'_d = \emptyset$, continue with step 6). Otherwise, choose a pair $(i, j) \in B'_d$ and remove it from B'_d.*
4) *Compute the S-vector of h_i and h_j and call it S_{ij}. Then compute the normal remainder $S'_{ij} = \mathrm{NR}_{\sigma,\mathcal{S}}(S_{ij})$. If $S'_{ij} = 0$, continue with step 3).*
5) *Increase s'' by one, append $h_{s''} = S'_{ij}$ to the tuple $\mathcal{S}$, append the set $\{(i, s'') \mid 1 \leq i < s'',\ \eta_i = \eta_{s''}\}$ to B', and continue with step 3).*
6) *If $B_d = \emptyset$, continue with step 10). Otherwise, choose a pair $(i, j) \in B_d$ and remove it from B_d.*
7) *Compute the S-vector of g_i and g_j and call it S_{ij}. Then compute the normal remainder $S'_{ij} = \mathrm{NR}_{\sigma,\mathcal{G}}(S_{ij})$. If S'_{ij} involves one of the indeterminates $e_1, \ldots, e_r$ then increase s' by one, append $g_{s'} = S'_{ij}$ to $\mathcal{G}$, append $\{(i, s') \mid 1 \leq i < s', \gamma_i = \gamma_{s'}\}$ to B, and continue with step 6).*

8) Compute the normal remainder $S''_{ij} = \mathrm{NR}_{\sigma,\mathcal{S}}(S'_{ij})$. *If* $S''_{ij} = 0$, *continue with step 6).*
9) Increase s'' *by one. Append* $h_{s''} = S''_{ij}$ *to the tuples* $\mathcal{S}$ *and* $\mathcal{S}_{\min}$. *Append* $\{(i, s'') \mid 1 \leq i < s'',\ \eta_i = \eta_{s''}\}$ *to* B'. *Continue with step 6).*
10) If $\mathcal{W}_d = \emptyset$, *continue with step 13). Otherwise, choose a vector* $v \in \mathcal{W}_d$ *and remove it from* $\mathcal{W}_d$.
11) Compute $v' = \mathrm{NR}_{\sigma,\mathcal{G}}(v)$ *and* $\bar{v} = v'(x_1, \ldots, x_n, e_1, \ldots, e_r, 0, \ldots, 0)$. *If* $\bar{v} = 0$, *continue with step 10).*
12) Increase s' *and* μ *by one. Adjoin a new indeterminate* ε_μ *to* $\overline{P}$ *and extend the grading to this new ring by defining* $\deg_{\overline{W}}(\varepsilon_\mu) = \deg_{\overline{W}}(v')$. *Extend the term ordering* σ *to the new ring in such a way that the extension is an elimination ordering for* $\{e_1, \ldots, e_r\}$. *Append* $g_{s'} = \bar{v} - \varepsilon_\mu$ *to the tuple* $\mathcal{G}$ *and* $\bar{v}$ *to* $\mathcal{G}_{\min}$. *Append* $\{(i, s') \mid 1 \leq i < s',\ \gamma_i = \gamma_{s'}\}$ *to* B. *Continue with step 10).*
13) If $B = B' = \emptyset$ *and* $\mathcal{W} = \emptyset$, *return the pair* $(\mathcal{G}_{\min}, \mathcal{S}_{\min})$ *and stop. Otherwise, continue with step 2).*

This is an algorithm which computes a pair $(\mathcal{G}_{\min}, \mathcal{S}_{\min})$ *where* $\mathcal{G}_{\min}$ *is a deg-ordered tuple of homogeneous vectors in* F_0 *which generate* M *minimally, and where* $\mathcal{S}_{\min}$ *is a deg-ordered tuple of homogeneous vectors in* $\oplus_{i=1}^{\mu} P(-d_{1i})$ *which generate* $\mathrm{Syz}_P(\mathcal{G}_{\min})$ *minimally. Here* μ *is the number of elements in* $\mathcal{G}_{\min}$ *and* d_{1i} *is the degree of the* i^{th} *element in* $\mathcal{G}_{\min}$ *for* $i = 1, \ldots, \mu$.

Proof. First we show finiteness. The procedure terminates when B, B', and $\mathcal{W}$ are empty. Only B and B' are enlarged during the computation. Each time the loop in steps 3)–5) is executed, one pair is removed from B_d (and thus from B). Similarly, performing the loops in steps 6)–9) and 10)–12) removes one element from B_d and $\mathcal{W}_d$, respectively. Hence it suffices to show that B and B' are enlarged only finitely many times.

The set B is enlarged in steps 7) and 12) only if an element $g_{s'}$ is appended to $\mathcal{G}$ whose normal remainder with respect to the previous elements of $\mathcal{G}$ is nonzero. In this case, $\langle \mathrm{LT}_\sigma(\mathcal{G}) \rangle$ is enlarged, and this can happen only finitely many times. Similarly, the set B' is enlarged in steps 5) and 9) only if an element $h_{s''}$ is appended to S which has a new leading term. Again this can happen only finitely many times.

Now we prove correctness. We consider the entire computation as a computation in $\widetilde{P} = K[x_1, \ldots, x_n, e_1, \ldots, e_r, \varepsilon_1, \ldots, \varepsilon_\mu]$. We claim that the elements of $\mathcal{W}$, $\mathcal{G}$, $\mathcal{G}_{\min}$, $\mathcal{S}$, and $\mathcal{S}_{\min}$ are always of the form $f_1 e_1 + \cdots + f_r e_r + \tilde{f}_1 \varepsilon_1 + \cdots + \tilde{f}_\mu \varepsilon_\mu$ with $f_i, \tilde{f}_j \in P$. This is clearly true at the outset. By the manner in which B and B' are enlarged in steps 5), 7), 9), and 12), the new S-vectors continue to have this shape. Furthermore, reducing one such vector using another one preserves the shape. Hence the elements of $\mathcal{G}$ and $\mathcal{S}$ have the desired form. Clearly the construction of $\mathcal{G}_{\min}$ in step 12) and $\mathcal{S}_{\min}$ in step 9) also preserves the given shape.

By induction on s'' it follows that elements $h_{s''}$ which are appended to $\mathcal{S}$ in step 5) and 9) do not involve the indeterminates $\{e_1, \ldots, e_r\}$. In particular, their leading position is an element $\eta_j \in \{1, \ldots, \mu\}$.

If we set $\varepsilon_1 \mapsto 0, \ldots, \varepsilon_\mu \mapsto 0$ in the entire algorithm and ignore the instructions concerning $\mathcal{S}$ and $\mathcal{S}_{\min}$, we see that we are applying Buchberger's Algorithm with Minimalization to the tuple $\mathcal{V}$. Therefore the tuple $\overline{\mathcal{G}} = (\bar{g}_1, \ldots, \bar{g}_{s'})$ where $\bar{g}_i =$

$g_i(x_1, \ldots, x_n, e_1, \ldots, e_r, 0, \ldots, 0)$ is a σ-Gröbner basis of M and $\mathcal{G}_{\min}$ is a subtuple of $\overline{\mathcal{G}}$ whose elements are a minimal system of generators of M.

Next we let $\mathcal{G}_{\min} = (\tilde{g}_1, \ldots, \tilde{g}_\mu)$. We claim that the elements of $\mathcal{G}$ map to zero under the homomorphism defined by $\varepsilon_1 \mapsto \tilde{g}_1, \ \ldots, \ \varepsilon_\mu \mapsto \tilde{g}_\mu$. This follows by induction on s' because in step 5) an element of $\langle g_1, \ldots, g_{s'-1} \rangle$ is appended to $\mathcal{G}$ and in step 8) we have

$$g_{s'}(x_1, \ldots, x_n, e_1, \ldots, e_r, \tilde{g}_1, \ldots, \tilde{g}_{\mu-1}) = \bar{v} - g_\mu = 0$$

Hence the elements of $\mathcal{G}$ are contained in the ideal $(\tilde{g}_1 - \varepsilon_1, \ldots, \tilde{g}_\mu - \varepsilon_\mu)$ of $\overline{P}$. Since we know already that $\mathcal{G}_{\min}$ is a minimal system of generators of M, it follows that $\tilde{I}_M = (\tilde{g}_1 - \varepsilon_1, \ldots, \tilde{g}_\mu - \varepsilon_\mu) + \widetilde{E}$ is the ideal of a homogeneous presentation of M. Therefore, by Proposition 3.3, it suffices to prove that $\mathcal{S} \cup \{\varepsilon_i \varepsilon_j \mid i, j = 1, \ldots, \mu\}$ is a σ-Gröbner basis of $\tilde{I}_M \cap P[\varepsilon_1, \ldots, \varepsilon_\mu]$ and $\mathcal{S}_{\min} \cup \{\varepsilon_i \varepsilon_j \mid i, j = 1, \ldots, \mu\}$ is a minimal set of generators of this ideal.

Since σ is an elimination ordering for $\{e_1, \ldots, e_r\}$, steps 3), 4), 5), 8), and 9) correspond to the application of Buchberger's Algorithm with Minimalization to this ideal. Hence the claim follows from Propositions 2.5.a and 2.8.a,b. Altogether, we have shown that $\mathcal{G}_{\min}$ is a minimal system of generators of M and $\mathcal{S}_{\min}$ is a minimal system of generators of $\mathrm{Syz}_P(\mathcal{G}_{\min})$. □

Let us add a few remarks about the inner workings of this algorithm.

Remark 4.5. Assume that we are in the setting of Theorems 4.2 and 4.4.

a) If we view the vertical and the horizontal strategy for computing a minimal homogeneous presentation as algorithms in $\overline{P}$, they are nothing but two concrete versions of the same general algorithm, namely the Homogeneous Buchberger Algorithm. The difference lies in the particular strategies for choosing the next pair to work on, and that we do not have to consider critical pairs (i, s') with $\gamma_i \neq \gamma_{s'}$ because $e_i\, e_{s'}$ maps to zero in the idealization of M.

b) The proof of Theorem 4.4 shows that when the algorithm stops the tuple $\overline{\mathcal{G}} = (\bar{g}_1, \ldots, \bar{g}_{s'})$ where $\bar{g}_i = g_i(x_1, \ldots, x_n, e_1, \ldots, e_r, 0, \ldots, 0)$ is a σ-Gröbner basis of M, and the tuple $\mathcal{S}$ is a σ-Gröbner basis of $\mathrm{Syz}_P(\mathcal{G}_{\min})$

c) Note that one can choose the term ordering on $\mathbb{T}^n\langle \varepsilon_1, \ldots, \varepsilon_\mu \rangle$ freely, as long as it extends σ. Hence the algorithm can be used to compute a Gröbner basis of the syzygy module $\mathrm{Syz}(\mathcal{G}_{\min})$ with respect to an arbitrary term ordering.

5. Computation of Minimal Graded Free Resolutions

As in the preceding sections, let K be a field, let $P = K[x_1, \ldots, x_n]$ be positively graded by $W \in \mathrm{Mat}_{m,n}(\mathbb{Z})$, let M be a graded P-submodule of a graded free P-module $F_0 = \oplus_{i=1}^{r_0}(-d_{0i})$, and let $\mathcal{V} = (v_1, \ldots, v_s)$ be a tuple of non-zero homogeneous vectors which generate M. For $d \in \mathbb{Z}^m$, the minimal graded free resolutions of M and $M(d)$ differ only by the shift d. Therefore we shall henceforth assume that we have $d_{0i} >_{\mathtt{Lex}} 0$ for $i = 1, \ldots, r_0$. In order to facilitate the formulation of our algorithms, we introduce a number of conventions.

The i^{th} graded free module F_i is always of the form $F_i = \oplus_{j=0}^{r_i}(-d_{ij})$. Its canonical basis is denoted by $\{\varepsilon_1^{(i)}, \ldots, \varepsilon_{r_i}^{(i)}\}$ and is always kept deg-ordered. The idealization of the module $F_0 \oplus \cdots \oplus F_n$ is a residue class ring of the ring $\widetilde{P} = P[\{\varepsilon_j^{(i)} \mid i \in \{0, \ldots, n\}, j \in \{1, \ldots, r_i\}\}]$. As in the previous section, the entire

computation can be viewed as a computation in $\widetilde{P}$. In particular, the elements of each F_i are identified with their images in this ring.

The first algorithm we present is a straightforward generalization of the vertical strategy for computing minimal presentations (see Theorem 4.2).

Theorem 5.1. (Computing Minimal Resolutions Vertically)
Let M be a graded submodule of a graded free P-module $F_0 = \oplus_{i=1}^{r_0} P(-d_{0i})$, where $d_{0i} >_{\mathtt{Lex}} 0$ for $i = 1, \ldots, r_0$. Let $\mathcal{V} = (v_1, \ldots, v_s)$ be a deg-ordered tuple of non-zero homogeneous vectors which generate M. Consider the following sequence of instructions.

1) *Let $i = 0$. Equip $\overline{P} = P[\varepsilon_1^{(i)}, \ldots, \varepsilon_{r_i}^{(i)}]$ with the grading defined by $\overline{W} = (W \mid d_{i1} \ \cdots \ d_{ir_i})$. Choose a term ordering σ on $\mathbb{T}^n(\varepsilon_1^{(i)}, \ldots, \varepsilon_{r_i}^{(i)})$. Let $B = \emptyset$, $\mathcal{W} = \mathcal{V}$, $\mathcal{G} = \emptyset$, $s' = 0$, $\mathcal{G}_{\min}^{(i)} = \emptyset$, $r_{i+1} = 0$, and $\mathcal{S} = \emptyset$.*
2) *Let d be the smallest degree with respect to $\mathtt{Lex}$ of an element in B or in $\mathcal{W}$. Form the subset B_d and the subtuple $\mathcal{W}_d$, and delete their entries from B and $\mathcal{W}$, respectively.*
3) *If $B_d = \emptyset$, continue with step 6). Otherwise, chose a pair $(j, k) \in B_d$ and remove it from B_d.*
4) *Form the S-vector S_{jk} of g_j and g_k. Then compute $S'_{jk} = \mathrm{NR}_{\sigma,\mathcal{G}}(S_{jk})$. If $S'_{jk} = 0$ continue with step 3). If $S'_{jk} \neq 0$ and it does not involve the indeterminates $\varepsilon_1^{(i)}, \ldots, \varepsilon_{r_i}^{(i)}$, then append it to $\mathcal{S}$ and continue with step 3).*
5) *Increase s' by one, append $g_{s'} = S'_{jk}$ to the tuple $\mathcal{G}$, and append the set $\{(j, s') \mid 1 \leq j < s', \ \gamma_j = \gamma_{s'}\}$ to B. Then continue with step 3).*
6) *If $\mathcal{W}_d = \emptyset$, continue with step 9). Otherwise, choose a vector $v \in \mathcal{W}_d$ and remove it from $\mathcal{W}_d$.*
7) *Compute $v' = \mathrm{NR}_{\sigma,\mathcal{G}}(v)$ and $\bar{v} = v'(x_1, \ldots, x_n, \varepsilon_1^{(i)}, \ldots, \varepsilon_{r_i}^{(i)}, 0, \ldots, 0)$. If $\bar{v} = 0$, continue with step 6).*
8) *Increase s' and r_{i+1} by one. Adjoin a new indeterminate $\varepsilon_{r_{i+1}}^{(i+1)}$ to $\overline{P}$ and extend the grading to this new ring by defining $\deg_{\overline{W}}(\varepsilon_{r_{i+1}}^{(i+1)}) = \deg_{\overline{W}}(\bar{v})$. Extend the term ordering σ to the new ring in such a way that the extension is an elimination ordering for $\{\varepsilon_1^{(i)}, \ldots, \varepsilon_{r_i}^{(i)}\}$. Append the element $g_{s'} = \bar{v} - \varepsilon_{r_{i+1}}^{(i+1)}$ to $\mathcal{G}$ and the element $\bar{v}$ to $\mathcal{G}_{\min}^{(i)}$. Append the set $\{(j, s') \mid 1 \leq j < s', \gamma_j = \gamma_{s'}\}$ to B. Continue with step 6).*
9) *If $B \neq \emptyset$ or $\mathcal{W} \neq \emptyset$, continue with step 2).*
10) *If $\mathcal{S} \neq 0$, then increase i by one and equip $\overline{P} = P[\varepsilon_1^{(i)}, \ldots, \varepsilon_{r_i}^{(i)}]$ with the grading defined by $\overline{W} = (W \mid d_{i1} \ \cdots \ d_{ir_i})$. Restrict the term ordering σ to $\mathbb{T}^n(\varepsilon_1^{(i)}, \ldots, \varepsilon_{r_i}^{(i)})$. Let $B = \emptyset$, $\mathcal{W} = \mathcal{S}$, $\mathcal{G} = \emptyset$, $s' = 0$, $\mathcal{G}_{\min}^{(i)} = \emptyset$, $r_{i+1} = 0$, and $\mathcal{S} = \emptyset$. Then continue with step 2).*
11) *Let $\ell = i + 1$. Return the list $(\mathcal{G}_{\min}^{(0)}, \ldots, \mathcal{G}_{\min}^{(\ell-1)})$ and stop.*

This is an algorithm which computes a list of deg-ordered homogeneous matrices $(\mathcal{G}_{\min}^{(0)}, \ldots, \mathcal{G}_{\min}^{(\ell-1)})$ such that the P-linear maps $\varphi_j : F_j \longrightarrow F_{j-1}$ given by $\mathcal{G}_{\min}^{(j-1)}$ for $j = 1, \ldots, \ell$ yield a minimal graded free resolution

$$0 \longrightarrow F_\ell \xrightarrow{\varphi_\ell} F_{\ell-1} \xrightarrow{\varphi_{\ell-1}} \cdots \xrightarrow{\varphi_3} F_2 \xrightarrow{\varphi_2} F_1 \xrightarrow{\varphi_1} M \longrightarrow 0$$

Proof. The loop defined by steps 2)–9) agrees with the main loop used in Theorem 4.2 to compute a minimal homogeneous presentation of M vertically. In the

proof of this theorem we saw that, after this loop is finished for some particular value of i, the tuple $\mathcal{G}_{\min}^{(i)}$ contains a minimal set of generators of the module generated by $\mathcal{W}$ and the tuple $\mathcal{S}$ contains a σ-Gröbner basis of $\mathrm{Syz}_P(\mathcal{G}_{\min}^{(i)})$. Thus, if we feed $\mathcal{S}$ as the new input tuple $\mathcal{W}$ into the next iteration of the loop, we compute a minimal system of generators $\mathcal{G}_{\min}^{(i+1)}$ of $\mathrm{Syz}_P(\mathcal{G}_{\min}^{(i)})$ and a Gröbner basis of $\mathrm{Syz}_P(\mathcal{G}_{\min}^{(i+1)})$. By Hilbert's Syzygy Theorem, this process stops after at most $n-1$ steps, i.e. after at most $n-1$ steps we have $\mathcal{S} = \emptyset$ and the algorithm stops. □

Although the vertical strategy for computing minimal graded free resolutions is in general rather crude, it has some merits.

Remark 5.2. Suppose we are in the setting of the theorem.

a) If we do not want to know the whole minimal graded free resolution of M, or if we discover during the computation that the going is getting too tough, we can stop the execution of the algorithm after the loop of steps 2)–9) is finished for a particular value of i and get the correct first $i+1$ graded free modules and maps.
b) When $i \geq 1$, we can save some applications of step 4), since we know that $\mathcal{S}$ is a Gröbner basis after the loop 2)–9) for $i-1$ is finished and this is the new input tuple $\mathcal{W}$.
c) In step 10) we can choose σ freely. Then we do not know that $\mathcal{W}$ is a Gröbner basis, i.e. the optimization of b) cannot be used, but we get in $\mathcal{G}$ a σ-Gröbner basis of $\mathrm{Syz}_P(\mathcal{G}_{\min}^{(i-1)})$ for the chosen term ordering σ.

Our second algorithm for computing minimal graded free resolutions generalizes the horizontal strategy for computing minimal homogeneous presentations (see Theorem 4.4). As before, we consider the entire computation as a computation in a large polynomial ring $\widetilde{P} = P[\varepsilon_1^{(0)}, \ldots, \varepsilon_{r_n}^{(n)}]$ a residue class ring of which is the idealization of $F_0 \oplus \cdots \oplus F_n$, and we identify all vectors with their images in $\widetilde{P}$. For the purposes of this theorem it will be convenient to replace the Division Algorithm (see for instance [7], 1.6.4) with a procedure that merely head reduces a given polynomial f with respect to a list of polynomials $\mathcal{G}$. We shall denote the result of such a reduction procedure by $\mathrm{HR}_{\sigma,\mathcal{G}}(f)$ and call it a **head reduction remainder** of f.

Theorem 5.3. (Computing Minimal Resolutions Horizontally)
Let M be a graded submodule of a graded free P-module $F_0 = \oplus_{i=1}^{r_0} P(-d_{0i})$, where $d_{0i} >_{\mathtt{Lex}} 0$ for $i = 1, \ldots, r_0$. Let $\mathcal{V} = (v_1, \ldots, v_s)$ be a deg-ordered tuple of non-zero homogeneous vectors which generate M. Consider the following sequence of instructions.

1) Let σ be a term ordering on $\mathbb{T}^n(\varepsilon_1^{(0)}, \ldots, \varepsilon_{r_0}^{(0)})$, let $\overline{P} = P[\varepsilon_1^{(0)}, \ldots, \varepsilon_{r_0}^{(0)}]$ be graded by $\overline{W} = (W \mid d_{01} \ \cdots \ d_{0r_0})$, let $r_1 = \cdots = r_n = 0$, let $B = \{v_1, \ldots, v_s\}$, let $\mathcal{G} = \emptyset$, and let $\mathcal{G}_{\min} = \emptyset$.
2) Let d be the smallest degree with respect to `Lex` *of an element of B. Form the subset B_d of B and remove it from B.*
3) If $B_d = \emptyset$, continue with step 7). Otherwise, let i be the largest upper index of an indeterminate $\varepsilon_k^{(j)}$ occurring in a polynomial of B_d. Let $f \in B_d$ be a polynomial which involves that indeterminate. Remove f from B_d.

4) Compute $f' = \mathrm{HR}_{\sigma,\mathcal{G}}(f)$. *If one of the indeterminates* $\varepsilon_1^{(i-1)}, \ldots, \varepsilon_{r_{i-1}}^{(i-1)}$ *occurs in* f', *append* f' *to* $\mathcal{G}$, *append to* B *all S-polynomials of* f' *and a polynomial* g *in* $\mathcal{G}$ *such that* $\mathrm{LT}_\sigma(f')$ *and* $\mathrm{LT}_\sigma(g)$ *involve the same indeterminate* $\varepsilon_j^{(i-1)}$, *and continue with step 3).*

5) If none of the indeterminates $\{\varepsilon_1^{(i)}, \ldots, \varepsilon_{r_i}^{(i)}\}$ *occurs in* f', *continue with step 3).*

6) Increase r_{i+1} *by one. Adjoin a new indeterminate* $\varepsilon_{r_{i+1}}^{(i+1)}$ *to* $\overline{P}$ *and extend the grading to this new ring by defining* $\deg_{\overline{W}}(\varepsilon_{r_{i+1}}^{(i+1)}) = d$. *Extend the term ordering* σ *to the new ring in such a way that the extension is an elimination ordering for* $\{\varepsilon_1^{(0)}, \ldots, \varepsilon_{r_i}^{(i)}\}$. *Compute the polynomial*

$$\overline{f} = f'(x_1, \ldots, x_n, \varepsilon_1^{(i)}, \ldots, \varepsilon_{r_i}^{(i)}, 0, \ldots, 0)$$

Append $g = \overline{f} - \varepsilon_{r_{i+1}}^{(i+1)}$ *to* $\mathcal{G}$ *and* $\overline{f}$ *to* $\mathcal{G}_{\min}$. *For all* $h \in B$ *such that* $\mathrm{LT}_\sigma(g)$ *and* $\mathrm{LT}_\sigma(h)$ *involve the same indeterminate* $\varepsilon_j^{(i)}$, *compute the S-polynomial of* g *and* h *and append it to* B. *Then continue with step 3).*

7) If $B = \emptyset$, *return the tuple* $\mathcal{G}_{\min}$ *and stop. Otherwise, continue with step 2).*

This is an algorithm which computes a deg-ordered tuple $\mathcal{G}_{\min}$ *of homogeneous polynomials in* $\overline{P} = P[\varepsilon_1^{(0)}, \ldots, \varepsilon_{r_0}^{(0)}, \ldots, \varepsilon_1^{(\ell)}, \ldots, \varepsilon_{r_\ell}^{(\ell)}]$ *such that the homogeneous maps of graded free* P*-modules* $\varphi_i : F_i \longrightarrow F_{i-1}$ *defined by the elements of* $\mathcal{G} \cap P[\varepsilon_1^{(i)}, \ldots, \varepsilon_{r_i}^{(i)}]$ *yield a minimal graded free resolution*

$$0 \longrightarrow F_\ell \xrightarrow{\varphi_\ell} F_{\ell-1} \xrightarrow{\varphi_{\ell-1}} \cdots \xrightarrow{\varphi_3} F_2 \xrightarrow{\varphi_2} F_1 \xrightarrow{\varphi_1} M \longrightarrow 0$$

Proof. First we prove that the procedure is well-defined, i.e. that all instructions can be executed. For this purpose it suffices to show the following claim: For a polynomial f which is contained in B or in $\mathcal{G}$ at some point during the computation, there exists a number $i \geq 0$ such that f is of the shape

$$f = a_1\varepsilon_1^{(i)} + \cdots + a_{r_i}\varepsilon_{r_i}^{(i)} + b_1\varepsilon_1^{(i+1)} + \cdots + b_{r_{i+1}}\varepsilon_{r_{i+1}}^{(i+1)}$$

with $a_j, b_k \in P$. This is clearly the case at the outset when $v_1, \ldots, v_s$ involve only the indeterminates $\varepsilon_1^{(0)}, \ldots, \varepsilon_{r_0}^{(0)}$. Later B and $\mathcal{G}$ are enlarged only in steps 4) and 6).

When $f' = \mathrm{HR}_{\sigma,\mathcal{G}}(f)$ is appended to $\mathcal{G}$ in step 4), the element $f \in B$ has been head reduced using only elements of the shape shown above. Thus also f' has this shape. When $g = \overline{f'} - \varepsilon_{r_{i+1}}^{(i+1)}$ is appended to $\mathcal{G}$ in step 6), the element f' involves only indeterminates $\varepsilon_k^{(j)}$ with upper index $j \in \{i, i{+}1\}$. Consequently, the polynomial $\overline{f'} = f'(x_1, \ldots, x_n, \varepsilon_1^{(i)}, \ldots, \varepsilon_{r_i}^{(i)}, 0, \ldots, 0)$ involves only indeterminates $\varepsilon_k^{(j)}$ with upper index $j = i$, and so g has the claimed shape. When B is enlarged in step 4) or 6), the new elements are S-polynomials of polynomials of the claimed shape. By the construction of σ, the leading term of a polynomial involves one of the indeterminates $\varepsilon_k^{(j)}$ with the smaller of the two upper indices. Since both leading terms are required to involve the same indeterminate $\varepsilon_k^{(j)}$, it follows that the S-polynomials appended to B have the claimed shape, and the claim is proved.

Now we turn to proving finiteness and correctness of the procedure. The procedure stops when B becomes empty. New S-polynomials are appended to B in step 4) or 6) only if a new element is appended to $\mathcal{G}$ in the same step. The new element of $\mathcal{G}$ has a leading term which is not contained in the ideal generated by $\mathrm{LT}_\sigma(\mathcal{G})$. Hence finiteness will follow if we can show that only finitely many new indeterminates $\varepsilon_{r_{i+1}}^{(i+1)}$ are introduced in step 6).

To prove this and the correctness of the algorithm, it is sufficient to prove the following claim by induction on i: After finitely many steps, all elements $f \in B$ involving indeterminates $\varepsilon_k^{(i)}$ with upper index i have been treated, and at that point the tuple $\mathcal{G}_i = \mathcal{G}_{\min} \cap P[\varepsilon_1^{(i)}, \ldots, \varepsilon_{r_i}^{(i)}]$ is a minimal system of generators of $\mathrm{Syz}(\mathcal{G}_{i-1})$. To this end, we set $\varepsilon_k^{(j)} \mapsto 0$ for $j > i$ everywhere and show that the resulting algorithm reduced to the horizontal strategy for computing minimal homogeneous presentations (see Theorem 4.4). The choice of f in step 3) ensures that S-polynomials involving only the indeterminates $\varepsilon_1^{(i)}, \ldots, \varepsilon_{r_i}^{(i)}$ are treated first. This corresponds to doing the loop in steps 3) – 5) before the loops in steps 6) – 9) and 10) – 12) in Theorem 4.4. The fact that we replaced the normal remainder computation by a head reduction remainder does not affect the correctness of the algorithm by Remark 2.5.6.a. The elements $f \in B$ which produce an f' appended to $\mathcal{G}$ in step 4) correspond to the pairs treated in steps 6) – 9) in Theorem 4.4. The elements f' discarded by step 5) correspond to the vectors v' discarded in step 11) in Theorem 4.4. Finally, step 6) corresponds to step 12) in Theorem 4.4. Altogether, it follows inductively from Theorem 4.4 that the algorithm is correct. By Hilbert's Syzygy Theorem, the tuple $\mathcal{G}_{\min}$ is finite and the computation stops after finitely many steps. □

The preceding algorithm admits several further optimizations.

Remark 5.4. Suppose that we are in the setting of the theorem.

a) In view of the checks preformed in steps 4) and 5), it suffices that the procedure $\mathrm{HR}_{\sigma,\mathcal{G}}(f)$ stops when all indeterminates $\varepsilon_k^{(i-1)}, \varepsilon_\ell^{(i)}$ are eliminated from f.
b) Let W be a matrix of non-negative integers. Then the head reduction procedure can be further optimized by noting that a polynomial f can be head reduced by $g \in \mathcal{G}$ only if $\deg_W(f)$ is componentwise larger than or equal to $\deg_W$. For instance, a homogeneous polynomial f of degree $\binom{4}{3}$ cannot be reduced by a homogeneous polynomial of degree $\binom{3}{4}$. This observation restricts the set of possible reductors in $\mathcal{G}$.

Aside from the simplicity of the formulation of the algorithm described in Theorem 4.4 (it requires merely seven steps!), it offers ample possibilities for introducing optimizations. For instance, the optimizations provided by M. Caboara and the authors in [4] can be used without difficulty. Although extensive practical tests are still lacking, the authors expect that this algorithm will provide a valuable tool for actual implementations.

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References

[1] J. Apel, A relationship between Gröbner bases of ideals and vector modules of G-algebras, In: Proc. Int. Conf. Novosibirsk 1989, Contemp. Math. **131**, Amer. Math. Soc., Providence 1992, pp. 195–204.
[2] A. Capani, G. De Dominicis, G. Niesi and L. Robbiano, Computing minimal resolutions, J. Pure Appl. Alg. **117-118** (1997), 105–117.

[3] M. Caboara, A modified Buchberger algorithm for resolutions, preprint 2003.
[4] M. Caboara, M. Kreuzer and L. Robbiano, Efficiently computing minimal sets of critical pairs, J. Symb. Comput. (to appear)
[5] M. Caboara and M. Silvestri, Classification of compatible module orderings, J. Pure Appl. Alg. **142** (1999), 13–24.
[6] M. Caboara and C. Traverso, The powerful Buchberger algorithm for modules, ??? (1997)
[7] M. Kreuzer and L. Robbiano, *Computational Commutative Algebra 1*, Springer, Heidelberg 2000.
[8] M. Kreuzer and L. Robbiano, *Computational Commutative Algebra 2*, Springer, Heidelberg 2004.
[9] M. Kreuzer and L. Robbiano, Basic tools for computing in multigraded rings, in: J. Herzog, V. Vuletescu (eds.), Commutative Algebra, Singularities and Computer Algebra, Proc. Conf. Sinaia 2002, NATO Science Series II, Kluwer Acad. Publ., Dordrecht 2003, pp. 197–216.
[10] R. La Scala and M. Stillman, Strategies for computing minimal free resolutions, J. Symb. Comput. **26** (1998), 409–431.
[11] M. Nagata, Local rings, Wiley & Sons, New York 1962.
[12] T. Siebert, Algorithms for the computation of free resolutions, in: B.H. Matzat, G.-M. Greuel and G. Hiss (eds.), Algorithmic Algebra and Number Theory, Springer, Berlin 1998, pp. 295–310.

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