Subharmonic methods in Banach algebra theory

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I declare that this essay is work done as part of the Part III Examination. It is the result of my own work, and except where stated otherwise, includes nothing which was performed in collaboration. No part of this essay has been submitted for a degree or any such qualification.

Signed ....

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Contents

Introduction

If A is a Banach algebra and x varies in A, what can be said about the spectrum function $x \mapsto \operatorname{Sp} x$? Is that function continuous or analytic in some sense? How behave related functions like the spectral radius function $x \mapsto \rho(x)$ or the spectral diameter function $x \mapsto \delta(x)$? Generally, the spectrum function does not have to be continuous, but one can relate it to so-called subharmonic functions; these are upper semicontinuous functions u from a domain D of $\mathbb{C}$ into $[-\infty, \infty)$ satisfying a submean inequality

$$u(z) \leq \frac{1}{2\pi} \int_0^{2\pi} u(z + re^{it})dt \qquad (\overline{D}(z,r) \subseteq D).$$

For instance, if $f : D \to A$ is an analytic function, then $\rho(f)$ and $\log \rho(f)$ are subharmonic functions. Subharmonic functions have a great number of beautiful properties, e. g. there is a maximum principle and a Liouville theorem for subharmonic functions. Using these we can prove some interesting results in Banach algebra theory very nicely.

We define subharmonic functions and develop their most important properties in chapter 1. In chapter 2 we discuss continuous and discontinuous behaviour of the spectrum and relate it to subharmonic functions. We then (chapter 3) give some applications where the subharmonicity of the spectrum is used to prove some results in Banach algebra theory. There is an appendix that deals with the radical of a Banach algebra and some basic facts from representation theory as neeeded in some of the applications in chapter 3.

The reader should have some general background in functional analysis. The basic facts needed about Banach algebras and spectral theory are gathered in section 2.1. I would like to thank Dr G R Allan for setting this very interesting essay and for his great support during the last months.

1 Subharmonic functions

We develop the theory of subharmonic functions on the complex domain based on the excellent book [5]. For a more general introduction to subharmonic functions see for instance [3].

1.1 Upper semicontinuous functions

As part of their definition, subharmonic functions are going to be upper semicontinuous, so we take a brief look at upper semicontinuous function in the abstract.

Definition 1.1.1 *Let (X, d) be a metric space. A function $u : X \to [-\infty, \infty)$ is called* upper semicontinuous *if, for every $c \in \mathbb{R}$, the set $\{x \in X : u(x) < c\}$ is an open subset in X.*

It is easy to check that u is upper semicontinuous if and only if

$$u(x) \geq \limsup_{y \to x} u(y) := \inf_{\varepsilon \to 0} \sup_{d(x,y) < \varepsilon} u(y) \qquad (x \in X).$$

It is also clear from the defintion that every upper semicontinuous function is a Borel function.

Proposition 1.1.2 *(a) If u_1, u_2 and u are upper semicontinuous functions and $\lambda \geq 0$, then $u_1 + u_2$, λu and $\max(u_1, u_2)$ are upper semicontinuous functions.*

(b) *If $\{u_i\}$ is a collection of upper semicontinuous functions, then $\inf_i u_i$ is upper semicontinuous. In particular, if $u_1 \geq u_2 \geq u_3 \ldots$ then $u(x) := \lim u_n(x)$ is upper semicontinuous.*

Proof. *(a)* Let $c \in \mathbb{R}$ and $U = \{x : \ u_1(x) + u_2(x) < c\}$. To show that U is open, let $x_0 \in U$ and $\varepsilon = c - u_1(x_0) + u_2(x_0) > 0$. Then $U_1 = \{x : \ u_1(x) < u_1(x_0) + \frac{\varepsilon}{2}\}$ and $U_2 = \{x : \ u_2(x) < u_2(x_0) + \frac{\varepsilon}{2}\}$ are neighbourhoods of x_0, hence $U_1 \cap U_2$ is a neighbourhood of x_0 contained in U.

Next, $\{x : \ \lambda u(x) < c\} = \{x : \ u(x) < \frac{c}{\lambda}\}$ for all $\lambda > 0$ and $c \in \mathbb{R}$, hence λu is upper semicontinuous if $\lambda > 0$ and trivially if $\lambda = 0$.

Furthermore, $\{x : \ \max(u_1(x), u_2(x)) < c\} = \{x : \ u_1(x) < c\} \cap \{x : u_2(x) < c\}$ is open for every $c \in \mathbb{R}$.

(b) If $c \in \mathbb{R}$, then $\{x : \ \inf_i u_i(x) < c\} = \cup_i \{x : \ u_i(x) < c\}$ is open. □

We shall make frequent use of the following basic compactness theorem.

Theorem 1.1.3 *If X is compact and u is an upper semicontinuous function on X, then u is bounded above on X and u attains its upper bound.*

Proof. The sets $(\{x : u(x) < n\})_{n \geq 1}$ form an open cover of X, so have a finite subcover. Hence u is bounded above on X. Let $M = \sup_X u$. Then the open sets $(\{x : u(x) < M - \frac{1}{n}\})_{n \geq 1}$ cannot cover X, because they have no finite subcover. Hence $u(x) = M$ for at least one $x \in X$. □

Now another preservation theorem can be stated.

Proposition 1.1.4 *Let X, T be metric spaces, T be compact and $v : X \times T \to [-\infty, \infty)$ be an upper semicontinuous function. Then $u : X \to [-\infty, \infty)$, $u(x) := \sup_{t \in T} v(x, t)$ is upper semicontinuous.*

Proof. By the preceding theorem $u(x) < \infty$ for all $x \in X$. To prove upper semicontinuity, let $c \in \mathbb{R}$ and $U = \{x \in X : \ u(x) < c\}$. If $x_0 \in U$, choose c' such that $u(x_0) < c' < c$. Then $v(x_0, t) < c'$ for each $t \in T$, hence by upper semicontinuity of v there are neighbourhoods M_t of x_0 and N_t of t such that $v < c'$ on $M_t \times N_t$. Now $T \subseteq N_{t_1} \cup \cdots \cup N_{t_n}$ for some $t_1, \ldots, t_n \in T$ by compactness, hence $M = M_{t_1} \cap \cdots \cap M_{t_n}$ is a neighbourhood of x_0 and we have $u(x) = \sup_{t \in T} v(x, t) \leq c' < c$ for $x \in M$. This shows that $M \subseteq U$ and therefore that U is open in X. □

The other result that we shall need is an approximation theorem.

Theorem 1.1.5 *If $u : X \to [-\infty, \infty)$ is upper semicontinuous and bounded above on X, then there is a decreasing sequence of uniformly continuous functions (f_n) on X such that for every x in X, $f_n(x) \downarrow u(x)$.*

Proof. We can suppose that u is not identically $-\infty$ (otherwise just take $f_n \equiv -n$). For $n \geq 1$, define $f_n : X \to \mathbb{R}$ by

$$f_n(x) = \sup_{y \in X}\{u(y) - nd(x, y)\} \qquad (x \in X)$$

where d is the metric on X. Since $|u(y) - nd(x, y) - (u(y) - nd(x', y))| = n|d(x, y) - d(x', y)| \leq nd(x, x')$, for each n, we have

$$|f_n(x) - f_n(x')| \leq nd(x, x') \qquad (x, x' \in X),$$

so f_n is uniformly continuous on X. Clearly also $f_1 \geq f_2 \geq \cdots \geq u$, and so in particular $\lim_{n\to\infty} f_n \geq u$. On the other hand, writing $B(x, r)$ for the ball $\{y \in X : \ d(x, y) < r\}$, we have

$$f_n(x) \leq \max(\sup_{B(x,r)} u, \ \sup_X u - nr) \qquad (x \in X, r > 0),$$

so that

$$\lim_{n\to\infty} f_n(x) \leq \sup_{B(x,r)} u \qquad (x \in X, r > 0).$$

As u is upper semicontinuous, letting $r \to 0$ gives $\lim_{n\to\infty} f_n \leq u$. □

1.2 Subharmonic functions

In analogy to convex functions on $\mathbb{R}$ we shall define subharmonic functions by a submean property (but here we have to assume upper semicontinuity as well).

Definition 1.2.1 *If D is a domain (a connected open subset) of $\mathbb{C}$, a function $u : D \to [-\infty, \infty)$ is* subharmonic *if u is upper semicontinuous and, for every closed disk $\overline{D}(z, r) \subseteq D$, we have the* submean inequality

$$u(z) \leq \frac{1}{2\pi}\int_0^{2\pi} u(z + re^{it})dt.$$

By Theorem 1.1.3 u is bounded above on $\partial D(z, r)$, so the integral is well-defined as an element of $[-\infty, \infty)$.

The usual definition assumes just the formally weaker (but equivalent) condition that the submean inequality holds just for all sufficiently small closed discs included in D. We do not need to use that formulation.

Example 1.2.2 *If $f : D \to \mathbb{C}$ is holomorphic, then both $|f|$ and $\log|f|$ are subharmonic on D.*

Proof. Both functions are continuous, so certainly upper semicontinuous. If $\overline{D}(z,r) \subseteq D$ then by Cauchy's integral formula

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} f(z+re^{it})dt, \text{ and so } |f(z)| \leq \frac{1}{2\pi}\int_0^{2\pi} |f(z+re^{it})|dt.$$

Thus $|f|$ is subharmonic on D.

For $\log|f|$ note first that the result is trivial if $f(z) = 0$. If $f(z) \neq 0$, we assume w.l.o.g. $z = 0$ and apply the classical Jensen's formula (see e. g. [7], Theorem 15.18):

Let f be holomorphic on $D(0,R)$, let $f(0) \neq 0$ and $0 < r < R$. Let the zeros of f in $\overline{D}(0,r)$ be $\alpha_1, \ldots, \alpha_N$, repeated according to multiplicity. Then,

$$\log|f(0)| = \frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{it})|dt - \log|\frac{r^N}{\alpha_1 \cdots \alpha_N}|.$$

□

Proposition 1.2.3 *(a) If u_1, u_2 and u are subharmonic functions and $\lambda \geq 0$, then $u_1 + u_2$, λu and $\max(u_1, u_2)$ are subharmonic.*

(b) If (u_n) is a decreasing sequence of subharmonic functions, then $u(z) := \lim u_n(z)$ is subharmonic.

(c) If (u_i) is a familiy of subharmonic functions and if $u(z) := \sup_i u_i(z)$ is upper semicontinuous, then u is subharmonic.

Proof. (a) These functions are upper semicontinuous by Proposition 1.1.2 (a) and one easily checks the submean inequality.

(b) By Proposition 1.1.2 (b) u is upper semicontinuous. Also, if $\overline{D}(a,r) \subseteq D$, then, for each n,

$$u(a) \leq u_n(a) \leq \frac{1}{2\pi}\int_0^{2\pi} u_n(a+re^{it})dt.$$

Now the monotone convergence theorem (note that u_1 is bounded above by Theorem 1.1.3) yields the desired result.

(c) By hypothesis u is upper semicontinuous and for each i we have

$$u_i(a) \leq \frac{1}{2\pi}\int_0^{2\pi} u_i(a+re^{it})dt \leq \frac{1}{2\pi}\int_0^{2\pi} u(a+re^{it})dt,$$

hence the result follows by taking the suprenum over i. $\square$

Now combining Proposition 1.1.4 with Proposition 1.2.3 (c) we immediately get the following result.

Proposition 1.2.4 *Let D be a domain in $\mathbb{C}$, T be a compact metric space and let $v : D \times T \to [-\infty, \infty)$ be upper semicontinuous such that $v(\cdot, t)$ is subharmonic on D for each $t \in T$, then $u : D \to [-\infty, \infty)$, $u(z) := \sup_{t\in T} v(z, t)$ is subharmonic.* $\square$

New examples of subharmonic functions are generated by composition with increasing convex functions.

Proposition 1.2.5 *Let $-\infty \leq a < b \leq \infty$, let $u : D \to [a, b)$ be a subharmonic function on a domain D and let $\psi : (a, b) \to \mathbb{R}$ be an increasing convex function. Then $\psi \circ u$ is subharmonic on D, where we define $\psi(a) = \lim_{t\to a} \psi(t)$.*

Proof. Choose $(a_n)_{n\geq 1} \in (a, b)$ with $a_n \downarrow a$ and for each n set $u_n = \max(u, a_n)$, so u_n is subharmonic. Since ψ is continuous and increasing, $\psi \circ u_n$ is upper semicontinuous on D. Also if $\overline{D}(z, r) \subseteq D$ then

$$\psi \circ u_n(z) \leq \psi\left(\frac{1}{2\pi}\int_0^{2\pi} u_n(z + re^{it})dt\right) \leq \frac{1}{2\pi}\int_0^{2\pi} \psi \circ u_n(z + re^{it})dt,$$

the second inequality coming from Jensen's inequality applied to the measure $dt/2\pi$ on $[0, 2\pi)$. Hence $\psi \circ u_n$ is subharmonic on U. Since $\psi \circ u_n \downarrow \psi \circ u$ as $n \to \infty$, it follows from Proposition 1.2.3 (b) that $\psi \circ u$ is subharmonic on U. $\square$

Corollary 1.2.6 *If u is subharmonic on a domain D of $\mathbb{C}$, then so is* $\exp u$. $\square$

For example, applying this result to $u := \alpha \log |f|$, where f is holomorphic and $\alpha > 0$, we deduce that $|f|^\alpha$ is subharmonic.

1.3 The maximum principle and some consequences

The maximum principle is our first important property of subharmonic functions. In the rest of this chapter we shall use this principle frequently.

Theorem 1.3.1 (maximum principle for subharmonic functions) *Let $u : D \to [-\infty, \infty)$ be a subharmonic function on a domain D of $\mathbb{C}$. If there exists $a \in D$ such that $u(z) \leq u(a)$ for all $z \in D$, then $u(z) = u(a)$ for all $z \in D$.*

Proof. Let $E = \{z \in D : \ u(z) = u(a)\} \neq \emptyset$, then E is closed since u is upper semicontinuous. If we show that E is open, then by connectedness of D it follows $E = D$ as required.

Now let $z \in E$ and $r > 0$ such that $D(z, r) \subseteq D$, then for $\rho < r$,

$$u(a) = u(z) \leq \frac{1}{2\pi} \int_0^{2\pi} u(z + \rho e^{it})dt,$$

but since $u(z + \rho e^{it}) \leq u(a)$ for each t, u must be equal to $u(a)$ almost everywhere on $\partial D(z, \rho)$ and by upper semicontinuity of u in fact everywhere, because $\{w \in \partial D(z, \rho) : \ u(w) < u(a)\}$ is open in $\partial D(z, \rho)$. Hence $u \equiv u(a)$ on $D(z, r)$ and E is indeed open. $\square$

For the next the result, we write $\partial_\infty S$ for the boundary of a subset $S \subseteq \mathbb{C}$ relative to the Riemann sphere $\mathbb{C}_\infty$, hence $\partial_\infty S = \partial S$ if S is bounded and $\partial_\infty S = \partial S \cup \{\infty\}$ if S is unbounded.

Corollary 1.3.2 *Let $u : D \to [-\infty, \infty)$ be a subharmonic function on a domain D and suppose there exists M such that $\limsup_{z\to a} u(z) \leq M$ for $a \in \partial_\infty D$. Then $u(z) \leq M$ for all $z \in D$.*

Proof. We extend u to $\partial_\infty D$ by defining $u(a) = \limsup_{z\to a} u(z)$, $a \in \partial_\infty D$. Then u is upper semicontinuous on $\overline{D}^\infty = D \cup \partial_\infty D$ which is compact, so by Theorem 1.1.3 u attains a maximum at some $z \in \overline{D}^\infty$. If $z \in \partial_\infty D$, then by assumption $u(z) \leq M$, so $u \leq M$ on D. On the other hand, if $z \in D$, then by the maximum principle u is constant on D, hence on $\overline{D}^\infty$, and so again $u \leq M$ on D. $\square$

Corollary 1.3.3 *If $u : D \to [-\infty, \infty)$ is a subharmonic function on a domain D, then for any $z \in D$, we have*

$$u(z) = \limsup_{w\to z, w\neq z} u(w).$$

Proof. By upper semicontinuity we have $u(z) \geq \limsup_{w\to z, w\neq z} u(w)$. Suppose $u(z) > \limsup_{w\to z, w\neq z} u(w) = \inf_{\varepsilon>0} \sup_{0<|z-w|<\varepsilon} u(w)$, then there exists $\varepsilon > 0$ such that $u(z) > u(w)$ for $0 < |z - w| < \varepsilon$. But then by the maximum principle u is constant on $D(z, \varepsilon)$ which is a contradiction. $\square$

It is sometimes also of interest to know under what conditions $\log u$ is subharmonic.

Theorem 1.3.4 *Let D be a domain of $\mathbb{C}$ and $u : D \to [0, \infty)$ be a function such that $u|e^p|$ is subharmonic for every polynomial p, then $\log u$ is subharmonic.*

Proof. If we take $p = 0$, we see that u is subharmonic and therefore upper semicontinuous, so that $\log u$ is upper semicontinuous.

To prove the submean inequality, let $\overline{D}(z, r) \subseteq D$ and $T := \partial D(z, r)$. By Theorem 1.1.5 we can choose continuous functions $f_n : T \to \mathbb{R}$ such that $f_n \downarrow \log u$ on T. By the Stone-Weierstrass theorem, for each $n \geq 1$, we can find a polynomial p_n such that

$$0 \leq \operatorname{Re} p_n - f_n \leq 1/n \text{ on } T.$$

Then we have

$$\limsup_{z \to a} u(z)|e^{-p_n(z)}| \leq e^{f_n(a)} e^{-\operatorname{Re} p_n(a)} \leq 1 \qquad (a \in T).$$

Since $u|e^{-p_n}|$ is assumed subharmonic, it follows from the maximum principle that $u|e^{-p_n}| \leq 1$ on $D(z, r)$, so in particular $\log u(z) \leq \operatorname{Re} p_n(z)$. Now $p_n(z) = \frac{1}{2\pi} \int_0^{2\pi} p_n(z + re^{it}) dt$ by Cauchy's formula, so we have

$$\begin{aligned} \log u(z) &\leq \operatorname{Re} p_n(z) = \frac{1}{2\pi} \int_0^{2\pi} \operatorname{Re} p_n(z + re^{it}) dt \\ &\leq \frac{1}{2\pi} \int_0^{2\pi} f_n(z + re^{it}) dt + \frac{1}{n}. \end{aligned}$$

Letting $n \to \infty$ and applying the monotone convergence theorem, we deduce that

$$\log u(z) \leq \frac{1}{2\pi} \int_0^{2\pi} \log u(z + re^{it}) dt$$

as desired. $\square$

1.4 Integrability

Although subharmonic functions are allowed to take the value $-\infty$, we shall see that non-constant subharmonic functions are almost everywhere finite.

We shall use the notion A for two-dimensional Lebesgue-measure.

Theorem 1.4.1 (integrability theorem) *Let u be a subharmonic function on a domain D of $\mathbb{C}$ being not identically $-\infty$. Then u is locally integrable on D, i. e. $\int_K |u|dA < \infty$ for each compact subset K of D.*

Proof. Let

$$A = \{z \in D : \exists r > 0 \text{ s. t. } \int_{D(z,r)} |u|dA < \infty\}$$

and $B = D \setminus A$. We show that A and B are open and that $u \equiv -\infty$ on B. Then we have $A = D$ and we use a simple compactness argument to finish the proof.

Now let $z \in A$ and $r > 0$ such that $\int_{D(z,r)} |u|dA < \infty$. We show that $D(z,r) \subseteq A$. If $a \in D(z,r)$, let $s := r - |z-a| > 0$ so that $D(a,s) \subseteq D(z,r)$ and therefore $\int_{D(a,s)} |u|dA \leq \int_{D(z,r)} |u|dA < \infty$, hence $a \in A$. Thus A is open.

If $z \in B$, choose $r > 0$ such that $\overline{D}(z, 2r) \subseteq D$. We show that $D(z,r) \subseteq B$ and that $u \equiv -\infty$ on $D(z,r)$. If $a \in D(z,r)$, let $s := r + |z-a|$ so that $D(a,s) \supseteq D(z,r)$. Now $\int_{D(a,s)} |u|dA \geq \int_{D(z,r)} |u|dA = \infty$ and u is bounded above on $\overline{D}(a,s)$, so we have $\int_{D(a,s)} udA = -\infty$. But u satisfies the submean inequality

$$u(a) \leq \frac{1}{2\pi} \int_0^{2\pi} u(a + \rho e^{it})dt \qquad (0 \leq \rho \leq s),$$

so by multiplying with $2\pi\rho$ and integrating we get

$$\int_0^s 2\pi\rho u(a)d\rho \leq \int_0^s \rho \int_0^{2\pi} u(a+\rho e^{it})dt = \int_{D(a,s)} udA = -\infty,$$

so that $u(a) = -\infty$. Hence $u \equiv -\infty$ on $D(z,r)$, so we have $D(z,r) \subseteq B$. Thus B is open and $u \equiv -\infty$ on B as required. □

Corollary 1.4.2 *Let u be a subharmonic function on a domain D of $\mathbb{C}$ being not identically $-\infty$. Then*

$$E := \{z \in D : u(z) = -\infty\}$$

is a set of Lebesgue measure zero.

Proof. Let $(K_n)_{n\geq 1}$ be compact sets with $\cup_n K_n = D$. For each n we have $\int_{K_n} |u|dA < \infty$, so $E \cap K_n$ has measure zero. Since $E = \cup_n (E \cap K_n)$, it too has measure zero. □

The set E above is also small in other ways. For example, E is totally disconnected, see e. g. [3], chapter 5.

1.5 Radial subharmonic functions

Before the next proof, note that harmonic functions are also subharmonic functions, because they are continuous and satisfy the mean equality $h(z) = \frac{1}{2\pi}\int_0^{2\pi} h(z+re^{it})dt$.

Theorem 1.5.1 *Let $v : D(0,\rho) \to [-\infty, \infty)$ be a subharmonic function being not identically $-\infty$ which is radial, that is $v(z) = v(|z|)$ for all z. Then $v(r)$ is an increasing convex function of $\log r$ $(0 < r < \rho)$.*

Proof. Let $r_1, r_2 \in (0,\rho)$ with $r_1 < r_2$ be given. The maximum principle applied to v on $D(0, r_2)$ yields

$$v(r_1) \le \sup_{\partial D(0,r_2)} v = v(r_2),$$

hence $v(r)$ is increasing.

Now v is locally integrable by Theorem 1.4.1, so it follows that $v(r) > -\infty$ for $r > 0$. We choose constants α, β such that $\alpha + \beta \log r = v(r)$ for $r = r_1, r_2$. Now $-\log|z|$ is harmonic, so $v(z) - \alpha - \beta \log|z|$ is subharmonic, hence applying the maximum principle on $\{z : \ r_1 < |z| < r_2\}$, we get

$$v(r) \le \alpha + \beta \log r \qquad (r_1 < r < r_2).$$

Hence, if $0 \le \lambda \le 1$ and $\log r = (1-\lambda)\log r_1 + \lambda \log r_2$, then

$$\begin{aligned} v(r) &\le \alpha + \beta \log r \\ &= (1-\lambda)(\alpha + \beta \log r_1) + \lambda(\alpha + \beta \log r_2) \\ &= (1-\lambda)v(r_1) + \lambda v(r_2), \end{aligned}$$

hence $v(r)$ is a convex function of $\log r$. $\square$

Remark: $\log r \mapsto f(r)$ is a convex function if and only if $t \mapsto f(e^t)$ is convex. For $0 \le \lambda \le 1$ we then have

$$f(s^\lambda t^{1-\lambda}) \le \lambda f(s) + (1-\lambda) f(t).$$

In particular, if $f = \log g$ for a non-negative function g, then

$$g(s^\lambda t^{1-\lambda}) \le g(s)^\lambda g(t)^{1-\lambda}.$$

Corollary 1.5.2 (Hadamard's three circles theorem) *Let u be a subharmonic function on $D(0,\rho)$ being not identically $-\infty$ and let $M_u(r) := \sup_{|z|=r} u(z)$ $(0 < r < \rho)$. Then $M_u(r)$ is an increasing convex function of $\log r$.*

Proof. For $0 < r < \rho$, we have $M_u(r) = v(r)$, where $v(z) = \sup_{t\in[0,2\pi]} u(ze^{it})$. Now v is subharmonic by Proposition 1.2.4 and obviously radial, so the result follows from the preceding theorem. □

Corollary 1.5.3 (Liouville's theorem for subharmonic functions) *Let $u : \mathbb{C} \to [-\infty, \infty)$ be a subharmonic function and suppose that $\liminf_{r\to\infty} M_u(r)/\log r = 0$. Then u is constant.*

Proof. If u is not identically $-\infty$, then we have $M(e^s) > -\infty$ for some $s \in \mathbb{R}$. By the preceding corollary $M(e^t)$ is increasing and convex, hence for $s < t$ we have

$$0 \leq \frac{M(e^t) - M(e^s)}{t - s} \leq \lim_{t\to\infty} \frac{M(e^t) - M(e^s)}{t - s} = \lim_{t\to\infty} \frac{M(e^t)}{t} = \lim_{r\to\infty} \frac{M(r)}{\log r} = 0.$$

Hence $M(e^t) = M(e^s)$ for $s < t$. Therefore u has a maximum inside $D(0, e^t)$, so the maximum principle forces u to be constant on $D(0, e^t)$ for every $t > 0$. Thus u is constant on $\mathbb{C}$. □

The following corollary is immediate.

Corollary 1.5.4 *Let $u : \mathbb{C} \to [-\infty, \infty)$ be subharmonic and bounded above. Then u is constant.* □

2 Analytic properties of the spectrum

In this chapter we discuss the behaviour of the spectrum function $x \mapsto \operatorname{Sp} x$. We give examples of continuity and non-continuity and we further develop the relations of the spectral radius and the spectral diameter to subharmonic functions. In this presentation we follow [2], section 3.4.

2.1 Prerequisites: Banach algebras and the spectrum

A *Banach algebra* is a complex Banach space $(A, \|\cdot\|)$ on which is defined an associative bilinear multiplication satisfying $\|ab\| \leq \|a\|\|b\|$ for all $a, b \in A$. Moreover, we always assume that A has a multiplicative identity $\mathbf{1}$ with $\|\mathbf{1}\| = 1$.

We shall write $G(A)$ for the set of all invertible elements of A, i. e. all elements $x \in A$ such that $yx = xy = \mathbf{1}$ for some $y \in A$. Given an element $x \in A$, its *spectrum* is defined by

$$\operatorname{Sp} x := \{\lambda \in \mathbb{C} : \ \lambda\mathbf{1} - x \notin G(A)\}$$

and its *spectral radius* by

$$\rho(x) := \sup\{|\lambda| : \ \lambda \in \operatorname{Sp} x\}.$$

We remind of some important facts:

Theorem 2.1.1 *Let A be a Banach algebra. We have:*

(a) If $x \in A$ and $\|x\| < 1$, then $1 + x \in G(A)$.

(b) $G(A)$ is open in A.

(c) $\operatorname{Sp} x$ is a non-empty compact set of $\mathbb{C}$, $x \in A$.

(d) $\lim \|x^n\|^{1/n} = \inf \|x^n\|^{1/n} = \rho(x)$; in particular, $\rho(x) \le \|x\|$, $x \in A$.

(e) If $x, y \in A$ such that $xy = yx$, then $\rho(x + y) \le \rho(x) + \rho(y)$ and $\rho(xy) \le \rho(x)\rho(y)$.

Proof. See e. g. [6], Theorems 10.7, 10.12, 10.13 and 11.23. □

Lemma 2.1.2 *Let A be a Banach algebra and let $x \in A$. Suppose (x_n) is a sequence of invertible elements converging to x such that (x_n^{-1}) is a bounded sequence. Then x is invertible.*

Proof. We have $\|1 - xx_n^{-1}\| = \|(x_n - x)x_n^{-1}\| \le \|x_n - x\|\|x_n^{-1}\| \to 0$ by assumption, so that xx_n^{-1} is invertible for large n. Hence x has a right inverse and similarly it can be proven that x has a left inverse. □

If A, B are Banach algebras (or just algebras), a linear operator $T : A \to B$ is called *homomorphism* if we have $T\mathbf{1} = \mathbf{1}$ and $T(xy) = TxTy$, for $x, y \in A$.

For the next theorem, we consider the algebra $\mathcal{O}(U)$ of holomorphic functions on an open set $U \subseteq \mathbb{C}$ with their (Fréchet-)topology of locally uniform convergence. We write $R(U)$ for the subalgebra of $\mathcal{O}(U)$ consisting of all rational functions having their poles in $\mathbb{C} \setminus U$.

Theorem 2.1.3 (holomorphic functional calculus) *Let A be a Banach algebra, $x \in A$ and $U \subseteq \mathbb{C}$ an open set containing $\operatorname{Sp} x$. Then there is a unique continuous homomorphism $\Theta_x : \mathcal{O}(U) \to A$ such that $\Theta_x(I) = x$ (where $I(\lambda) := \lambda$). Moreover:*

(a) If γ is an arbitrary cycle that encloses $\operatorname{Sp} x$ in U, we have

$$\Theta_x(f) = \frac{1}{2\pi i}\int_\gamma f(\lambda)(\lambda\mathbf{1} - x)^{-1} d\lambda.$$

(b) *For* $r \in R(U)$, *we have* $\Theta_x(r) = r(x)$.

(c) $\operatorname{Sp}\Theta_x(f) = f(\operatorname{Sp} x)$, *for all* $f \in \mathcal{O}(U)$.

With respect to (b) we shall write $f(x)$ *for* $\Theta_x(f)$, *for all* $f \in \mathcal{O}(U)$.

Proof. See e. g. [2], Theorem 3.3.3. □

The following application of the holomorphic functional calculus will be useful quite often.

Proposition 2.1.4 *Let* A *be a Banach algebra. Suppose that* $x \in A$ *and that* $\alpha \notin \operatorname{Sp} x$. *Then we have*

$$1/\operatorname{dist}(\alpha, \operatorname{Sp} x) = \rho((\alpha\mathbf{1} - x)^{-1}).$$

Proof. Let $U \supseteq \operatorname{Sp} x$ be open such that $\alpha \notin U$. Then $f(\lambda) = 1/(\alpha - \lambda)$ is holomorphic on U. So we have $\operatorname{Sp}(\alpha\mathbf{1} - x)^{-1} = \{1/(\alpha - \lambda) : \ \lambda \in \operatorname{Sp} x\}$ by Theorem 2.1.3 (c) and in particular,

$$\begin{aligned} \rho((\alpha\mathbf{1} - x)^{-1}) &= \sup\{1/|\alpha - \lambda| : \ \lambda \in \operatorname{Sp} x\} \\ &= 1/\inf\{|\alpha - \lambda| : \ \lambda \in \operatorname{Sp} x\} = 1/\operatorname{dist}(\alpha, \operatorname{Sp} x). \end{aligned}$$

□

2.2 Properties concerning the continuity

For compact subsets K_1, K_2 of $\mathbb{C}$, define the *Hausdorff distance* by

$$\Delta(K_1, K_2) = \max(\sup_{z \in K_2} \operatorname{dist}(z, K_1), \sup_{z \in K_1} \operatorname{dist}(z, K_2)).$$

We shall say that $x \mapsto \operatorname{Sp} x$ is *continuous at* $a \in A$ if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that $||x - a|| < \delta$ implies $\Delta(\operatorname{Sp} x, \operatorname{Sp} a) < \varepsilon$.

If A is a commutative Banach algebra, then the spectrum function is continuous. In fact, we have a more precise result:

Theorem 2.2.1 *Let* A *be a Banach algebra. Suppose that* $x, y \in A$ *commute. Then* $\operatorname{Sp} y \subseteq \operatorname{Sp} x + \overline{D}(0, \rho(x - y))$ *and consequently we have* $\Delta(\operatorname{Sp} x, \operatorname{Sp} y) \leq \rho(x - y) \leq \|x - y\|$.

Proof. Suppose the inclusion is not true. This means that there exists $\alpha \in \operatorname{Sp} y$ such that $\operatorname{dist}(\alpha, \operatorname{Sp} x) > \rho(x-y)$. Therefore by Proposition 2.1.4 we have $\rho((\alpha\mathbf{1}-x)^{-1})\rho(x-y) < 1$. Since $(\alpha\mathbf{1}-x)^{-1}$ and $x-y$ commute, we have $\rho((\alpha\mathbf{1}-x)^{-1}(x-y)) \leq \rho((\alpha\mathbf{1}-x)^{-1})\rho(x-y) < 1$. But then $\alpha\mathbf{1}-y = (\alpha\mathbf{1}-x)(1+(\alpha\mathbf{1}-x)^{-1}(x-y))$ is invertible, contradicting $\alpha \in \operatorname{Sp} y$. □

In general, the spectrum function need not to be continuous:

Example 2.2.2 *Let H be the separable Hilbert space and let A be the Banach algebra $\mathcal{L}(H)$. There exists $T \in A$ and a sequence (T_k) in A such that $T_k \to T$ and $\operatorname{Sp}(T_k) = \{0\}$ for all $k \geq 1$, but $\operatorname{Sp} T \neq \{0\}$.*

Proof. Let (α_n) be the sequence of positive numbers defined by $\alpha_n = e^{-k}$ if $n = 2^k(2l+1)$ (so that k counts the number of prime factors 2 of n). If (e_n) is an orthonormal basis of H, define $T \in A$ by $Te_n = \alpha_n e_{n+1}$ and $T_k \in A$ for $k \geq 1$ by

$$T_k e_n = \begin{cases} 0 & \text{if } n = 2^k(2l+1) \text{ for some } l \\ \alpha_n e_{n+1} & \text{otherwise.} \end{cases}$$

Now we have $\|T - T_k\| \leq e^{-k}$ so that $T_k \to T$. Furthermore one easily checks $T_k^{2^{k+1}} e_n = 0$ for every n, so that T_k is nilpotent and hence $\operatorname{Sp} T_k = \{0\}$.

We show that $\rho(T) > 0$ so that $\operatorname{Sp} T \neq \{0\}$. Now $T^m e_1 = \alpha_1 \cdots \alpha_m e_{m+1}$, hence $\|T^m\| \geq \alpha_1 \cdots \alpha_m$. By the definition of the sequence (α_n) we have $\alpha_1 \ldots \alpha_{2^t-1} = \prod_{j=0}^{t-1} \exp(-j2^{t-j-1})$, $t \geq 1$, and therefore

$$\begin{aligned} (\alpha_1 \ldots \alpha_{2^t-1})^{1/(2^t-1)} &= \prod_{j=0}^{t-1} \exp(-j2^{t-j-1})^{1/2^t-1} > \prod_{j=0}^{t-1} \exp(-j2^{t-j-1})^{1/2^{t-1}} \\ &= \prod_{j=0}^{t-1} \exp(-j2^{-j}) = \exp(-\sum_{j=0}^{t-1} j2^{-j}) \geq \exp(-\sigma), \end{aligned}$$

where $\sigma := \sum_{j=0}^{\infty} j2^{-j}$. Hence $\rho(T) = \lim \|T^m\|^{1/m} \geq e^{-\sigma} > 0$ as required. □

Although the spectrum function might be discontinuous, we have the following result.

Theorem 2.2.3 *Let A be a Banach algebra. Then the spectrum function $x \mapsto \operatorname{Sp} x$ is upper semicontinuous on A, that is, for every open set U containing $\operatorname{Sp} x$ there exists $\delta > 0$ such that $\|x - y\| < \delta$ implies $\operatorname{Sp} y \subset U$. In particular, the spectral radius ρ is upper semicontinuous on A.*

Proof. Let $U \supset \operatorname{Sp} x$ be open and suppose the theorem not to be true. Then there exist sequences (x_n) and (α_n) such that $x_n \to x$ and $\alpha_n \in \operatorname{Sp} x_n \cap (\mathbb{C} \setminus U)$. Then $|\alpha_n| \leq \|x_n\|$ so that (α_n) is bounded. By the Bolzano-Weierstrass theorem, we may assume that $\alpha_n \to \alpha$ for some $\alpha \in \mathbb{C}$. Because U is open, we have $\alpha \notin U$, hence $\alpha\mathbf{1} - x$ is invertible. But the set of invertible elements is open, so that $\alpha_n\mathbf{1} - x_n$ is invertible for large n, contradicting $\alpha_n \in \operatorname{Sp} x_n$. $\square$

The first important results concerning spectral variation are due to J. D. Newburgh. First a lemma:

Lemma 2.2.4 *Let A be a Banach algebra and let $x \in A$. Let $U \supset \operatorname{Sp} x$ be an open set and let $f : U \to \mathbb{C}$ be a holomorphic function. If (x_n) is a sequence in A such that $x_n \to x$, we have $\operatorname{Sp} x_n \subset U$ for large n and $f(x_n) \to f(x)$.*

Proof. Choose a bounded open set V such that $\operatorname{Sp} x \subset V \subset \overline{V} \subset U$. Then we have $\operatorname{Sp} x_n \subset V$ for large n by Theorem 2.2.3. We assume w.l.o.g. that $\operatorname{Sp} x_n \subset V$ for all $n \geq 1$.

Now $\overline{V}$ is a compact subset of U, so there exists a cycle γ in U that encloses $\overline{V}$ and therefore $\operatorname{Sp} x$ and all $\operatorname{Sp} x_n$, $n \geq 1$. By the Theorem 2.1.3 (a) we have

$$2\pi i(f(x_n) - f(x)) = \int_\gamma f(\lambda)[(\lambda\mathbf{1} - x_n)^{-1} - (\lambda\mathbf{1} - x)^{-1}]d\lambda.$$

Since $\operatorname{Im}\gamma$ is compact, to prove $f(x_n) \to f(x)$ it suffices to show

$$\sup_{\lambda \in \operatorname{Im}\gamma} \|(\lambda\mathbf{1} - x_n)^{-1} - (\lambda\mathbf{1} - x)^{-1}\| \to 0.$$

Now $(\lambda\mathbf{1}-x_n)[(\lambda\mathbf{1}-x_n)^{-1}-(\lambda\mathbf{1}-x)^{-1}](\lambda\mathbf{1}-x) = (\lambda\mathbf{1}-x)-(\lambda\mathbf{1}-x_n) = x_n-x$, so that

$$\begin{aligned} \|(\lambda\mathbf{1} - x_n)^{-1} - (\lambda\mathbf{1} - x)^{-1}\| &= \|(\lambda\mathbf{1} - x_n)^{-1}(x_n - x)(\lambda\mathbf{1} - x)^{-1}\| \\ &\leq \sup_{\lambda,n} \|(\lambda\mathbf{1} - x_n)^{-1}\|\|x_n - x\| \sup_{\lambda} \|(\lambda\mathbf{1} - x)^{-1}\|, \end{aligned}$$

where both suprema are finite because $\operatorname{Im}\gamma$ and $\{x_n : \ n \geq 0\} \cup \{x\}$ are compact. Since $x_n \to x$ the proof is finished. $\square$

Theorem 2.2.5 (J. D. Newburgh) *Let A be a Banach algebra and let $x \in A$. Suppose U, V are disjoint open sets such that $\operatorname{Sp} x \subset U \cup V$ and $\operatorname{Sp} x \cap U \neq \emptyset$. Then there exists $r > 0$ such that $\|x - y\| < r$ implies $\operatorname{Sp} y \cap U \neq \emptyset$.*

Proof. By Theorem 2.2.3, there exists $\delta > 0$ such that $\|x - y\| < \delta$ implies $\operatorname{Sp} y \subset U \cup V$. If the theorem is false, there exists a sequence (x_n) such that $x_n \to x$ and $\operatorname{Sp} x_n \subset V$ for $n \geq 1$.

Let f be the holormophic function on $U \cup V$ defined by 1 on U and 0 on V. Now if $n \geq 1$ and γ_n is a circle that encloses $\operatorname{Sp} x_n$ in V, we have $2\pi i f(x_n) = \int_{\gamma_n} f(\lambda)(\lambda \mathbf{1} - x)^{-1} d\lambda = 0$ by Theorem 2.1.3 (a), so that $f(x_n) = 0$. On the other hand, $\operatorname{Sp} f(x) = f(\operatorname{Sp} x)$ contains 1 so that $f(x) \neq 0$. But we have $f(x_n) \to f(x)$ by the preceding lemma, a contradiction. $\square$

Corollary 2.2.6 (J. D. Newburgh) *Let A be a Banach algebra, $a \in A$ and suppose that $\operatorname{Sp} a$ is totally disconnected. Then $x \mapsto \operatorname{Sp} x$ is continuous at a.*

Proof. If $\operatorname{Sp} a$ is totally disconnected, then for every $\varepsilon > 0$, there are some $\lambda_i \in \mathbb{C}$ and $\varepsilon_i \leq \varepsilon$ such that $\operatorname{Sp} a$ is contained in a disjoint union $D(\lambda_1, \varepsilon_1) \cup \cdots \cup D(\lambda_n, \varepsilon_n)$ and such that $\operatorname{Sp} a \cap D(\lambda_i, \varepsilon_i) \neq \emptyset$ for $1 \leq i \leq n$. Now by Theorem 2.2.3 and Theorem 2.2.5 there exists $\delta > 0$ such that $\|x - a\| < \delta$ implies

$$\operatorname{Sp} x \subset D(\lambda_1, \varepsilon_1) \cup \cdots \cup D(\lambda_n, \varepsilon_n) \text{ and } \operatorname{Sp} x \cap D(\lambda_i, \varepsilon_i) \neq \emptyset, 1 \leq i \leq n.$$

Hence $\operatorname{dist}(\lambda, \operatorname{Sp} a) \leq 2\varepsilon$ for $\lambda \in \operatorname{Sp} x$ and $\operatorname{dist}(\lambda, \operatorname{Sp} x) \leq 2\varepsilon$ for $\lambda \in \operatorname{Sp} a$, so that $\Delta(\operatorname{Sp} x, \operatorname{Sp} a) \leq 2\varepsilon$. $\square$

This implies in particular that the spectral function is continuous at all elements having finite or countable spectrum.

Corollary 2.2.7 *Let A be a Banach algebra and let $k \geq 1$. Then*

$$B_k := \{a \in A : \# \operatorname{Sp} a \leq k\}$$

is a closed set in A.

Proof. Let $a \in A$ and (a_n) be a sequence in B_k converging to a. Suppose $\# \operatorname{Sp} a > k$, so that $\{\lambda_1, \ldots, \lambda_{k+1}\} \subseteq \operatorname{Sp} a$ for some λ_i. Now choose $\varepsilon > 0$ such that $D(\lambda_1, \varepsilon) \cup \cdots \cup D(\lambda_{k+1}, \varepsilon)$ is a disjoint union. By Theorem 2.2.5 we then have $\operatorname{Sp} a_n \cap D(\lambda_i, \varepsilon) \neq \emptyset$ for $1 \leq i \leq k+1$ and large n, so that $\# \operatorname{Sp} a_n > k$, a contradiction. $\square$

2.3 Subharmonic properties of the spectrum

Lemma 2.3.1 *Let f be an analytic function from a domain D of $\mathbb{C}$ into a Banach space X. Then $\lambda \mapsto \log\|f(\lambda)\|$ is subharmonic on D.*

Proof. Clearly, $\log\|f\|$ is continuous, hence upper semicontinuous. If B is the closed unit ball of X^*, then

$$\log\|f(\lambda)\| = \sup\{\log|(\phi\circ f)(\lambda)| : \ \phi\in B\}.$$

Since $\phi\circ f$ is holomorphic for every ϕ, $\log|\phi\circ f|$ is subharmonic by Example 1.2.2, hence Proposition 1.2.3 (c) yields the result. □

This is the main theorem which connects subharmonic functions with Banach algebras:

Theorem 2.3.2 (E. Vesentini) *Let f be an analytic function from a domain D of $\mathbb{C}$ into a Banach algebra A. Then both $\lambda \mapsto \rho(f(\lambda))$ and $\lambda \mapsto \log\rho(f(\lambda))$ are subharmonic functions on D.*

Proof. Using Theorem 1.2.5, we only have to show that $\lambda \mapsto \log\rho(f(\lambda))$ is subharmonic. For $n \geq 1$ set

$$u_n(z) = 2^{-n}\log\|f(z)^{2^n}\| \qquad (z\in U).$$

Since f^{2^n} is a holomorphic function, the previous lemma implies that u_n is subharmonic on U. Also, because $\|a^{2^{n+1}}\| \leq \|a^{2^n}\|\|a^{2^n}\|$ for all $a\in A$, the sequence (u_n) is decreasing, and by the spectral readius formula it converges to $\log\rho(f)$. Hence by Proposition 1.2.3 (b) $\log\rho(f)$ is subharmonic on D. □

Note that this result yields again the upper semicontinuity of the spectral radius ρ, without using Theorem 2.2.3.

Like the spectral radius, the spectral diameter has subharmonic behaviour, too:

Theorem 2.3.3 *Let f be an analytic function from a domain D of $\mathbb{C}$ into a Banach algebra A. Then the functions $\lambda\mapsto\delta(f(\lambda))$ and $\lambda\mapsto\log\delta(f(\lambda))$ are both subharmonic, where $\delta(x) := \operatorname{diam}\operatorname{Sp}(x) := \sup\{|\lambda-\mu| : \ \lambda,\mu\in\operatorname{Sp}x\}$ for $x\in A$.*

Proof. Let $x\in A$ and $\alpha\in\mathbb{C}$ with $|\alpha|=1$. We denote by $\delta_\alpha(x)$ the length of the projection of $\operatorname{Sp}x$ on the line $\{t\overline{\alpha} : \ t\in\mathbb{R}\}$. We then have $\delta(x) = \sup_{|\alpha|=1}\delta_\alpha(x)$ and

$$\delta_\alpha(x) = \sup\{\operatorname{Re}(\alpha\lambda) : \ \lambda\in\operatorname{Sp}x\} - \inf\{\operatorname{Re}(\alpha\lambda) : \ \lambda\in\operatorname{Sp}x\}.$$

If we consider the holomorphic function $\lambda \mapsto e^{\alpha\lambda}$ on $\mathbb{C}$, by the holomorphic functional calculus we have $\operatorname{Sp} e^{\alpha x} = e^{\alpha \operatorname{Sp} x}$ so that

$$\log \rho(e^{\alpha x}) = \log \sup\{|e^{\alpha\lambda}| : \ \lambda \in \operatorname{Sp} x\} = \sup\{\operatorname{Re}(\alpha\lambda) : \ \lambda \in \operatorname{Sp} x\}.$$

Similarly, $\log \rho(e^{-\alpha x}) = -\inf\{\operatorname{Re}(\alpha\lambda) : \ \lambda \in \operatorname{Sp} x\}$ and thus we have

$$\delta_\alpha(x) = \log \rho(e^{\alpha x}) + \log \rho(e^{-\alpha x}).$$

Therefore, $\delta_\alpha(f)$ is subharmonic by Vesentini's theorem for each $|\alpha| = 1$. Hence $\delta(f) = \sup_{|\alpha|=1} \delta_\alpha(f)$ is subharmonic by Proposition 1.2.4, noting that $(\lambda, \alpha) \mapsto \delta_\alpha(f(\lambda))$ is upper semicontinuous by Theorem 2.2.3.

Now if p is a polynomial, then $\lambda \mapsto e^{p(\lambda)} f(\lambda)$ is analytic so that $\delta(e^p f) = |e^p|\delta(f)$ is subharmonic by the first part. We now use Theorem 1.3.4 to conclude that $\log \delta(f)$ is subharmonic. $\square$

Remark: Using Vesentini's theorem, there are several other analytic properties of the spectrum to discover (see [2], section 3.4). Let $f : D \to A$ be an analytic function on a domain D of $\mathbb{C}$ into a Banach algebra A and consider the function $\lambda \mapsto \operatorname{Sp} f(\lambda)$. We then have e. g. a spectral maximum principle and a spectral Liouville theorem. Also one can prove that isolated spectral values are holomorphic functions.

3 Applications

We now give some examples where subharmonicity is used in Banach algebra theory. Some of them use the notion of the radical of a Banach algebra and some use representation theory, where we refer to the appendix in both cases. I follow [2], chapter 5, although I changed some of the proofs.

3.1 Some elementary applications

Theorem 3.1.1 *Let a be an element of a Banach algebra and let U be an open bounded set containing* $\operatorname{Sp} a$. *Then* $\sup_{\lambda \in \partial U} \rho((a - \lambda\mathbf{1})^{-1}(x - a)) < 1$ *implies* $\operatorname{Sp} x \subseteq U$.

Proof. Let $D = \mathbb{C} \setminus \overline{U}$, then $a - \lambda\mathbf{1}$ is invertible for $\lambda \in D$, so that

$$f(\lambda) = (a - \lambda\mathbf{1})^{-1}(x - a)$$

is a well-defined analytic function on the domain D. Now, by hypothesis $C := \sup_{\lambda \in \partial U} \rho((a - \lambda\mathbf{1})^{-1}(x - a)) < 1$, so we have

$$\limsup_{\lambda \to \lambda_0} \rho(f(\lambda)) \leq C \text{ for } \lambda_0 \in \partial D = \partial U \text{ and}$$

$$\rho(f(\lambda)) \leq \|f(\lambda)\| \to 0 \text{ for } \lambda \to \infty.$$

Hence by the maximum principle (Corollary 1.3.2), we have $\rho(f(\lambda)) \leq C < 1$ for all $\lambda \in \overline{D}$. So $1 + f(\lambda)$ is invertible for $\lambda \in \overline{D} = \mathbb{C} \setminus U$ and hence

$$x - \lambda\mathbf{1} = (a - \lambda\mathbf{1})(a - \lambda\mathbf{1})^{-1}(x - a + a - \lambda\mathbf{1}) = (a - \lambda\mathbf{1})(f(\lambda) + 1)$$

is invertible as well. Therefore $\lambda \notin \operatorname{Sp} x$ for $\lambda \in \mathbb{C} \setminus U$ and the theorem is proved. □

As an application we prove a theorem due to Geršgorin dealing with nearly diagonal matrices.

Corollary 3.1.2 (S. A. Geršgorin) *Let $x = (a_{ij})$ be a $n \times n$ matrix and let $a = \operatorname{diag}(a_{11}, \ldots, a_{nn})$. Suppose that for some $\varepsilon > 0$, we have $\|x - a\| < \varepsilon$, where $\|\cdot\|$ is the $\mathcal{L}(l_p^n)$-norm for some $1 \leq p \leq \infty$, and that $D(a_{ii}, \varepsilon)$ and $D(a_{jj}, \varepsilon)$ have disjoint or identical boundaries for $i \neq j$. Then $\operatorname{Sp} x \subseteq D(a_{11}, \varepsilon) \cup \cdots \cup D(a_{nn}, \varepsilon)$.*

Proof. Let $U := D(a_{11}, \varepsilon) \cup \cdots \cup D(a_{nn}, \varepsilon)$, then clearly $\operatorname{Sp} a \subseteq U$, so by Theorem 3.1.1 it remains to check that $\sup_{\lambda \in \partial U} \rho((a - \lambda\mathbf{1})^{-1}(x - a)) < 1$. But by hypothesis, $\lambda \in \partial U$ implies $|\lambda - a_{ii}| \geq \varepsilon$ for all i, so we have

$$\|(a - \lambda\mathbf{1})^{-1}\| = \|\operatorname{diag}(1/(a_{11} - \lambda), \ldots, 1/(a_{nn} - \lambda)\| = \max_i |1/(a_{ii} - \lambda)| \leq 1/\varepsilon,$$

so that

$$\sup_{\lambda \in \partial U} \rho((a - \lambda\mathbf{1})^{-1}(x - a)) \leq \sup_{\lambda \in \partial U} \|(a - \lambda\mathbf{1})^{-1}\| \cdot \|x - a\| < 1/\varepsilon \cdot \varepsilon = 1$$

as required. □

For the next application, we need a lemma.

Lemma 3.1.3 *Let A be a Banach algebra and let $a, b \in A$. Then we have*

$$e^a b e^{-a} = \sum_n \frac{1}{n!} \underbrace{[a, \ldots, [a}_{n \text{ times}}, b]...],$$

where $[a, b] := ab - ba$ is the commutator *of a and b.*

Proof. We have

$$e^a b e^{-a} = \sum_n \sum_{k+l=n} \frac{a^k}{k!} \cdot b \cdot \frac{(-a)^l}{l!} = \sum_n \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} a^k b (-a)^{n-k},$$

these series converging absolutely, so it suffices to prove

$$\sum_{k=0}^{n} \binom{n}{k} a^k b(-a)^{n-k} = \underbrace{[a, \ldots, [a}_{n \text{ times}}, b] \ldots] \text{ for each } n,$$

what can be done by a simple induction. □

Theorem 3.1.4 (D. C. Kleinecke-F. V. Širokov) *Let a, b be in a Banach algebra. Suppose that $a(ab-ba) = (ab-ba)a$. Then $\rho(ab-ba) = 0$.*

Proof. Using the previous lemma with the hypothesis $[a, [a, b]] = 0$, we get

$$e^{\lambda a} b e^{-\lambda a} = \sum_n \frac{\lambda^n}{n!} \underbrace{[a, \ldots, [a}_{n \text{ times}}, b] \ldots] = b + \lambda [a, b]$$

for all $\lambda \in \mathbb{C}$. Define the subharmonic function $u(\mu) = \rho(\mu b + [a, b])$ on $\mathbb{C}$, then by Theorem 1.3.3 we have

$$\rho([a, b]) = u(0) = \limsup_{\mu \to 0, \mu \neq 0} u(\mu),$$

where $u(\mu) = |\mu| \rho(b + [a, b]/\mu) = |\mu| \rho(e^{\frac{1}{\mu} a} b e^{-\frac{1}{\mu} a}) = |\mu| \rho(b) \to 0$ for $\mu \to 0$, $\mu \neq 0$ (we have used the elementary fact that $\operatorname{Sp} yxy^{-1} = \operatorname{Sp} x$ for invertible y). Hence $\rho([a, b]) = 0$. □

Theorem 3.1.5 *Let a, b be in a Banach algebra and suppose there exists $R > 0$ such that $\operatorname{Sp} a \cap D(0, R)$ has no accumulation point other than 0 (this holds for example if a is a compact linear operator). If $(ab-ba)a = 0$ or $a(ab-ba) = 0$, then $\rho(ab-ba) = 0$.*

Proof. Suppose for instance that $(ab-ba)a = 0$, the other case being studied similarly. Then $(b - \frac{1}{\lambda}(ab-ba))(\lambda \mathbf{1} - a) = \lambda b - (ab-ba) - ba + 0 = (\lambda \mathbf{1} - a) b$ for all $\lambda \neq 0$. Hence for $\lambda \notin \operatorname{Sp} a \cup \{0\}$, we have $(\lambda \mathbf{1} - a) b (\lambda \mathbf{1} - a)^{-1} = b - \frac{1}{\lambda}(ab-ba)$, so that $\rho(b) = \rho(b - \frac{1}{\lambda}(ab-ba))$ and

$$|\lambda| \rho(b) = \rho(\lambda b - (ab-ba)).$$

Now consider the subharmonic function $u(\lambda) = \log \rho(\lambda b - (ab-ba))$ on $\mathbb{C}$. For every $r > 0$, we have the submean inequality

$$u(0) \leq \frac{1}{2\pi} \int_0^{2\pi} u(re^{it}) dt$$

and if $re^{it} \notin \operatorname{Sp} a$, we have

$$u(re^{it}) = \log(|re^{it}| \rho(b)) = \log r + \log \rho(b).$$

But by hypothesis, if $0 < r < R$, $re^{it} \notin \operatorname{Sp} a$ for almost all t, hence we have $u(0) \leq \log r + \log \rho(b)$. Now letting $r \to 0$ yields $u(0) = -\infty$ and therefore $\rho(ab-ba) = 0$. □

3.2 Spectral characterizations of commutative Banach algebras

If A is commutative, we know that the spectral radius is subadditive, submultiplicative and uniformly continuous on A. By Proposition A.1.3 (b) the same result is also true supposing $A/\operatorname{Rad} A$ to be commutative. Surprisingly, the converse is true. (The definition of the radical $\operatorname{Rad} A$ and some basic facts can be found in the appendix A.1.)

Definition 3.2.1 *For a Banach algebra A, we define the* centre modulo the radical

$$Z(A) = \{a \in A : \ ax - xa \in \operatorname{Rad} A, x \in A\}.$$

Lemma 3.2.2 *Let $a \in A$ be such that $\# \operatorname{Sp}(ax - xa) = 1$ for all $x \in A$. Then $a \in Z(A)$.*

Proof. (This proof uses representation theory; an introduction can be found in the appendix A.2.) If $\# \operatorname{Sp}(ax - xa) = 1$ for all $x \in A$, then by Lemma A.3.6, for each continuous irreducible representation π of A, we have $\pi(a) = \alpha \mathbf{1}$ for some $\alpha \in \mathbb{C}$, so that $\pi(ax - xa) = \alpha\pi(x) - \pi(x)\alpha = 0$. Hence $ax - xa \in \ker \pi$ for all π and so $ax - xa \in \operatorname{Rad} A$ by Proposition A.2.5. $\square$

Theorem 3.2.3 *Let $a \in A$. Then the following properties are equivalent:*

(a) $a \in Z(A)$,

(b) there exists $M > 0$ such that $\rho(a + x) \le M(1 + \rho(x))$, for every $x \in A$,

(c) there exists $N > 0$ such that $\rho((a - \lambda\mathbf{1})^{-1}x) \le N\rho((a - \lambda\mathbf{1})^{-1})\rho(x)$, for every $x \in A$ and $\lambda \notin \operatorname{Sp} a$.

Proof. (a)$\Rightarrow$(b),(c). If we consider $\tilde{A} = A/\operatorname{Rad} A$, then $\rho_A(x) = \rho_{\tilde{A}}(\tilde{x})$ by Proposition A.1.3 (b). A straightforward check yields also $a \in Z(A)$ if and only if $\tilde{a} \in Z(\tilde{A})$. Therefore we can replace A by $\tilde{A}$ and hence (by Proposition A.1.3 (a)) assume w.l.o.g. that A is semi-simple.

Then (a) implies that a commutes with every $x \in A$, hence $\rho(a + x) \le \rho(a) + \rho(x)$ and (b) holds with $M = \max(1, \rho(a))$. Moreover, $(a - \lambda\mathbf{1})^{-1}$ commutes with every $x \in A$ as well, so that by the submultiplicity of the spectral radius (c) holds with $N = 1$.

(b)$\Rightarrow$(a). We will show that $\rho(au - ua) = 0$ for every $u \in A$ and use Lemma 3.2.2 to conclude that $a \in Z(A)$. Fix $u \in A$ and define

$$f(\lambda) = \begin{cases} \frac{1}{\lambda}(a - e^{\lambda u} a e^{-\lambda u}) & \text{for } \lambda \neq 0 \\ [a, u] & \text{for } \lambda = 0. \end{cases}$$

Then f is analytic on $\mathbb{C}$, because by Lemma 3.1.3 we have

$$\frac{1}{\lambda}(a - e^{\lambda u} a e^{-\lambda u}) = \frac{1}{\lambda}(-\lambda[u,a] - \frac{\lambda^2}{2}[u,[u,a]] - \dots) \to [a,u] \text{ for } \lambda \to 0.$$

Now by hypothesis, for $\lambda \neq 0$, we have $\rho(f(\lambda)) \leq \frac{M}{|\lambda|}(1 + \rho(e^{\lambda u} a e^{-\lambda u})) = \frac{M}{|\lambda|}(1 + \rho(a))$. So the subharmonic function $\lambda \mapsto \rho(f(\lambda))$ tends to zero at infinity. By the maximum principle it is identically 0 and consequently $\rho(au - ua) = \rho(f(0)) = 0$ as desired.

(c)$\Rightarrow$(b). If we look at the proof of Theorem 2.2.1, we conclude that (c) implies $\operatorname{Sp} y \subseteq \operatorname{Sp} a + D(0, N\rho(y-a))$, for every $y \in A$. In particular we have $\rho(a+x) \leq \rho(a) + N\rho(x) \leq M(1+\rho(x))$ for $M = \max(\rho(a), N)$. $\square$

Corollary 3.2.4 *Let A be a Banach algebra. Then the following properties are equivalent:*

- *(a) $A/\operatorname{Rad} A$ is commutative.*
- *(b) ρ is subadditive on A, that is there exists $M > 0$ such that $\rho(x+y) \leq M(\rho(x) + \rho(y))$, for all $x, y \in A$.*
- *(c) ρ is submultiplicative on A, that is there exists $N > 0$ such that $\rho(xy) \leq N\rho(x)\rho(y)$, for all $x, y \in A$.*
- *(d) ρ is uniformly continous on A, which implies that there exists $C > 0$ such that $|\rho(x) - \rho(y)| \leq C\|x - y\|$, for all $x, y \in A$.*

Proof. The equivalence of (a), (b) and (c) is clear from Theorem 3.2.3 and we have (a)$\Rightarrow$(d) by Theorem 2.2.1. If (d) holds, then $\rho(a+x) \leq C\|a\| + \rho(x)$, for every $x, a \in A$, hence we can use Theorem 3.2.3 (b)$\Rightarrow$(a) to conclude (a). $\square$

3.3 Automatic continuity for Banach algebra homomorphisms

If A is a Banach algebra and $T : A \to B$ is a surjective homomorphism onto a semi-simple Banach algebra B (semi-simplicity is defined in the appendix A.1), then T is automatically continuous. This is a famous result proved 1967 by B. E. Johnson using mainly representation theory. We now give a simple proof of this result using subharmonic functions.

Note that if T is a homomorphism, then T maps the invertible elements of A into invertible elements of B. Hence $\operatorname{Sp} Tx \subseteq \operatorname{Sp} x$ and therefore $\rho(Tx) \leq$

$\rho(x) \leq \|x\|$. In fact, it suffices to ask for a surjective linear mapping satisfying $\rho(Tx) \leq \|x\|$ to conclude its continuity.

We first need a lemma and we shall call $q \in A$ *quasi-nilpotent* if $\rho(q) = 0$.

Lemma 3.3.1 *Let A be a Banach algebra. Let $a \in A$ and suppose $\rho(a+q) = 0$ for all quasi-nilpotent elements $q \in A$. Then $a \in \operatorname{Rad} A$.*

Proof. Taking $q = 0$, we have $\rho(a) = 0$ and so $\rho(e^u a e^{-u}) = 0$ for all $u \in A$, therefore $\rho(a - e^u a e^{-u}) = 0$. Let $f : \mathbb{C} \to A$ be the analytic function defined as in Theorem 3.2.3, then $\rho(f(\lambda))$ is a subharmonic function with $\rho(f(\lambda)) = 0$ for $\lambda \neq 0$. Hence by Theorem 1.3.3, we have

$$\rho(au - ua) = \rho(f(0)) = \limsup_{\lambda \to 0, \lambda \neq 0} \rho(f(\lambda)) = 0.$$

Hence, by Lemma A.3.6, for every continuous irreducible representation π, we have $\pi(a) = \alpha \mathbf{1}$ for some $\alpha \in \mathbb{C}$. But $\alpha = \rho(\pi(a)) \leq \rho(a) = 0$ (note that π is a homomorphism), so that $a \in \ker \pi$. Hence $a \in \operatorname{Rad} A$ by Proposition A.2.5. □

Theorem 3.3.2 *Let A and B be two Banach algebras with B semi-simple. Suppose that T is a surjective linear mapping from A onto B such that $\rho(Tx) \leq \|x\|$, for every $x \in A$. Then T is continuous.*

Proof. Let (a_n) be a sequence in A such that $a_n \to 0$ and $Ta_n \to b$ in B. By the closed graph theorem, it will follow that T is continuous if we can show that $b = 0$. We will show that $\rho(b + y) = 0$ for every quasi-nilpotent element y of B. Then Theorem 3.3.1 forces b to be in $\operatorname{Rad} B = \{0\}$ and hence $b = 0$ as desired.

So let a quasi-nilpotent $y \in B$ be given. For $z \in B$ define

$$f_z(\lambda) = (\lambda - 1)z + b + y,$$

then $\log \rho(f_z(\lambda))$ is a subharmonic function on $\mathbb{C}$. Let $M : B \times (0, \infty) \to \mathbb{R}$ be defined as $M(z, r) := \sup_{|\lambda| = r} \rho(f_z(\lambda))$, then Hadamard's three circle theorem (Corollary 1.5.2, with the preceding remark) states that

$$\rho(b + y)^2 = \rho(f_z(1))^2 \leq M(z, r) \cdot M(z, 1/r).$$

In particular, for a sequence (z_n) in B, we get

$$\rho(b + y)^2 \leq \limsup_{n \to \infty} M(z_n, r) \cdot \limsup_{n \to \infty} M(z_n, 1/r).$$

Now, $M(z,r) = \sup_{t\in[0,2\pi]} \rho(f_z(re^{it}))$ and because $(t,z,r) \mapsto \rho(f_z(re^{it}))$ is upper semicontinuous on $[0,2\pi]\times B\times(0,\infty)$, M is also upper semicontinuous by Proposition 1.1.4. Hence considering $z_n = Ta_n \to b$, we get

$$\limsup_{n\to\infty} M(z_n, r) \leq M(b,r) = \sup_{|\lambda|=r} \rho(\lambda b + y).$$

Now choose $a, x \in A$ such that $Ta = b$ and $Tx = y$. Using the assumption $\rho(Tu) \leq \|u\|$ for $u \in A$, we get on the other hand

$$\begin{aligned} M(z_n, 1/r) &= \sup_{|\lambda|=1/r} \rho(T(\lambda a_n + a + x - a_n)) \\ &\leq \|a_n\|/r + \|a + x - a_n\|, \end{aligned}$$

so that $\limsup_{n\to\infty} M(z_n, 1/r) \leq \|a + x\|$ since $a_n \to 0$. Inserting these estimates, we get

$$\rho(b+y)^2 \leq \sup_{|\lambda|=r} \rho(\lambda b + y) \cdot \|a + x\|$$

for $r > 0$. Hence letting $r \to 0$ and using again the upper semicontinuity of ρ, we finally get $\rho(b+y)^2 \leq \rho(y) \cdot \|a + x\| = 0$ as desired. □

Corollary 3.3.3 (B. E. Johnson) *Let A and B be two Banach algebras, with B semi-simple. Suppose that T is a surjective homomorphism from A onto B. Then T is continuous.* □

As a consequence, we see that the algebraic and topological structures in a Banach algebra are tied together much more closely than one might suspect from the original definition:

Corollary 3.3.4 (Uniqueness-of-norm Theorem) *Let $(A, \|\cdot\|)$ be a semi-simple Banach algebra and let $|||\cdot|||$ be another norm making $(A, |||\cdot|||)$ to a Banach algebra. Then $\|\cdot\|$ and $|||\cdot|||$ are equivalent.*

Proof. Because the definition of semi-simplicity is purely algebraic, we see that $(A, |||\cdot|||)$ is semi-simple as well. If $j : (A, \|\cdot\|) \to (A, |||\cdot|||)$ is the identity-map, j and j^{-1} are both continuous by Johnson's theorem, so that $\|\cdot\|$ and $|||\cdot|||$ are equivalent. □

3.4 Spectral characterizations of finite-dimensional algebras

Let A be a Banach algebra such that $A/\operatorname{Rad} A$ is finite-dimensional. For all $x \in A$, the class $\tilde{x}$ is algebraic in $A/\operatorname{Rad} A$ and consequently $\operatorname{Sp} x$ is finite. Surprisingly, the converse is true even supposing that the spectrum is finite only on a non-empty open set of A.

We will use the following extension of Theorem 2.3.3. The proof is rather complicated and we refer to [2], Theorems 7.1.3 and 7.1.13.

Theorem 3.4.1 *Let f be an analytic function from a domain D of $\mathbb{C}$ into a Banach algebra A. Then for arbitrary $n \geq 1$ the functions $\lambda \mapsto \delta_n(f(\lambda))$ and $\lambda \mapsto \log \delta_n(f(\lambda))$ are subharmonic on D, where δ_n denotes the n-*th spectral diameter *defined by*

$$\delta_n(x)^{n(n+1)/2} = \sup\{ \prod_{1 \leq i < j \leq n+1} |\lambda_i - \lambda_j| : \ \lambda_1, \ldots, \lambda_{n+1} \in \operatorname{Sp} x \}.$$

□

We have the following corollaries (which depend in the case $n = 1$ only on the proven Theorem 2.3.3, of course):

Corollary 3.4.2 *Let f be an analytic function from a domain D of $\mathbb{C}$ into a Banach algebra A. Let $n \geq 1$ and let $E = \{\lambda \in D : \ \# \operatorname{Sp} f(\lambda) \leq n\}$ (being closed in D by Corollary 2.2.7). If E has positive measure, we have $E = D$.*

Proof. By Theorem 3.4.1 $u := \log \delta_n(f)$ is subharmonic and we have $E = \{\lambda \in D : u = -\infty\}$, because $\delta_n(x) = 0$ if and only if $\# \operatorname{Sp} x \leq n$. But if u is not identically $-\infty$, E is a set of measure zero by Corollary 1.4.2, contradicting the assumption. Hence $u \equiv -\infty$ and $E = D$. □

Corollary 3.4.3 *Let A be a Banach algebra and $U \subseteq A$ be a non-empty open set.*

(a) If there exists $n \geq 1$ such that $\# \operatorname{Sp} x \leq n$ for all $x \in U$, then we have $\# \operatorname{Sp} x \leq n$ for all $x \in A$.

(b) If $\operatorname{Sp} x$ is finite for all $x \in U$ (so that n_x may depend on x), then there exists $n \geq 1$ such that $\# \operatorname{Sp} x \leq n$ for all $x \in A$.

Proof. (a) Choose $a \in U$, let $x \in A$ and consider the analytic function $f(\lambda) = a + \lambda(x - a)$ on $\mathbb{C}$. Now $E = \{\lambda \in \mathbb{C} : \# \operatorname{Sp} f(\lambda) \leq n\}$ has non-empty interior, and hence is a set of positive measure. By the preceding Corollary we then have $E = \mathbb{C}$ and in particular $1 \in E$, so that $\# \operatorname{Sp} x = \# \operatorname{Sp} f(1) \leq n$.

(b) Let $B_k = \{x \in A : \# \operatorname{Sp} x \leq k\}$, then $B = \cup_k B_k \supseteq U$, so that B has non-empty interior. Now the sets B_k are closed by Corollary 2.2.7, so by Baire's theorem there exists $n \geq 1$ such that B_n has non-empty interior. We then apply (a) to conclude $B_n = A$. $\square$

If we connect this Corollary with a result mentioned in the appendix, then we get our main theorem for this section.

Theorem 3.4.4 *Let A be a Banach algebra and $U \subseteq A$ be a non-empty open set such that $\operatorname{Sp} x$ is finite for all $x \in U$. Then $A / \operatorname{Rad} A$ is finite-dimensional.*

Proof. Replacing A by $A / \operatorname{Rad} A$ does not affect the spectrum, hence we may assume w.l.o.g. that A is semi-simple. Now Corollary 3.4.3 (b) and Theorem A.3.4 yield the desired result. $\square$

We now give an application of Corollary 3.4.3 (a) (with $n = 1$).

Theorem 3.4.5 *Let A be a Banach algebra containing a non-empty open set U of invertible elements such that $\rho(x)\rho(x^{-1}) = 1$ for all $x \in U$. Then $A / \operatorname{Rad} A \cong \mathbb{C}$.*

Proof. If we replace A by $A / \operatorname{Rad} A$, then the spectrum is not affected, so we can w.l.o.g assume that A is semi-simple.

Let $x \in U$. There exists $r > 0$ such that $|\lambda| < r$ implies $x - \lambda \mathbf{1} \in U$ and therefore $\rho(x - \lambda\mathbf{1})\rho(x - \lambda\mathbf{1})^{-1} = 1$. By Proposition 2.1.4 we then have $\rho(x - \lambda\mathbf{1}) = \operatorname{dist}(\lambda, \operatorname{Sp} x)$, so that

$$\operatorname{Sp} x \subseteq \overline{D}(\lambda, \operatorname{dist}(\lambda, \operatorname{Sp} x)), \qquad |\lambda| < r.$$

We claim that $\# \operatorname{Sp} x = 1$. Suppose there are distinct $\mu, \nu \in \operatorname{Sp} x$ and suppose $|\mu| \leq |\nu|$. Then there exists λ with $|\lambda| < r$ and $|\lambda - \mu| < |\lambda - \nu|$. But then $\operatorname{dist}(\lambda, \operatorname{Sp} x) \leq |\lambda - \mu| < |\lambda - \nu|$ so that $\nu \notin \overline{D}(\lambda, \operatorname{dist}(\lambda, \operatorname{Sp} x))$ contradicting $\nu \in \operatorname{Sp} x$. Thus $\# \operatorname{Sp} x = 1$ for all $x \in U$, hence by Corollary 3.4.3 (a) we have $\# \operatorname{Sp} x = 1$ for all $x \in A$.

We now prove $A \cong \mathbb{C}$. If $x \in A$, then $\# \operatorname{Sp} x = \{\lambda\}$ for some $\lambda \in \mathbb{C}$. If π is a continuous irreducible representation on a Banach space X, we have $\dim X = 1$ by Lemma A.3.2. So we have $\emptyset \neq \operatorname{Sp} \pi(x) \subseteq \operatorname{Sp} x = \{\lambda\}$, so that

$\pi(x) = \lambda\mathbf{1}$. This being true for all continuous irreducible representation on X, we conclude $x = \lambda\mathbf{1}$. Hence $A \cong \mathbb{C}$. $\square$

We can even drop the assumption of semi-simplicity in the next interesting result, which shows that in non-trivial Banach algebras there always exists an invertible element $x \in A$ such that $\|x^{-1}\| > 1/\|x\|$.

Corollary 3.4.6 *Let A be a Banach algebra containing a non-empty open set U of invertible elements such that $\|x\| \cdot \|x^{-1}\| = 1$, for all $x \in U$. Then $A \cong \mathbb{C}$.*

Proof. For $x \in U$, we have $1 = \rho(1) \leq \rho(x)\rho(x^{-1}) \leq \|x\| \cdot \|x^{-1}\| = 1$, so $A/\operatorname{Rad} A \cong \mathbb{C}$ by the preceding theorem. But we show that A is in fact semi-simple.

First, Let $G_1(A)$ be the component in the set of the invertible elements that contains $\mathbf{1}$ and we assume w.l.o.g. that U is connected. Let

$$B := \{x \in G_1(A) : \ \|x\| \cdot \|x^{-1}\| = 1\}.$$

Then B is a closed subset of $G_1(A)$ containing $\mathbf{1}$. We claim that B is open. Choose $a \in U$, then Ua^{-1} is a neighbourhood of $\mathbf{1}$. If $x \in B$ and $y \in Ua^{-1}$, then we have $xy \in B$ because $1 \leq \|xy\|\|y^{-1}x^{-1}\| = \|xyaa^{-1}\|\|aa^{-1}y^{-1}x^{-1}\| \leq \|x\|\|ya\|\|a^{-1}\|\|a\|\|a^{-1}y^{-1}\|\|x^{-1}\| = 1$ and xUa^{-1} is a connected set containing x. Hence $xUa^{-1} \subseteq B$. Thus B is open, so that $B = G_1(A)$.

Now we can show that $\operatorname{Rad} A = \{0\}$. Suppose $x \in \operatorname{Rad} A$ and $x \neq 0$. For $t \geq 0$, $1 + tx$ is invertible (Proposition A.1.2) and therefore $x_t = \frac{1+tx}{1+t} \in G_1(A)$, so that $\|x_t\|\|x_t^{-1}\| = 1$. Now if $t \to \infty$, we have $x_t \to x$ and therefore $\|x_t^{-1}\| = 1/\|x_t\| \to 1/\|x\|$. Then x is invertible by Lemma 2.1.2, contradicting $\rho(x) = 0$. Hence $\operatorname{Rad} A = \{0\}$ and $A \cong \mathbb{C}$ as required. $\square$

A Appendix

This appendix deals with the radical of a Banach algebra and some representation theory as needed in chapter 3. The material presented here is basically from [1] and [2].

A.1 The radical of a Banach algebra

Let A be a Banach algebra. A linear subspace $I \subseteq A$ is a *left ideal* if $xI \subseteq I$ for all $x \in A$. It is called *maximal* if, for every proper left-ideal (i. e. $J \subset A$,

$J \neq A$) containing I, we have $J = I$. Similarly, (maximal) *right ideals* and *two-sided ideals* are defined.

In this appendix we will use the fact that every proper left ideal is contained in a maximal left ideal and that every maximal left ideal is closed (see e. g. [2], Lemma 3.1.1 and Corollary 3.2.2).

Definition A.1.1 *The* radical $\operatorname{Rad} A$ *of a Banach algebra A is defined as the intersection of all maximal left ideals of A. If* $\operatorname{Rad} A = \{0\}$, *then we say that A is* semi-simple.

An example of a semi-simple Banach algebra is the algebra $\mathcal{L}(X)$ of all bounded linear operators on a Banach space (see e. g. [2], Theorem 3.1.4).

The next proposition provides alternative formulations for the radical.

Proposition A.1.2 *Let A be a Banach algebra. Then the following statements are equivalent:*

(a) $x \in I$ for all maximal left ideals I of A.

(b) $x \in J$ for all maximal right ideals J of A.

(c) $\mathbf{1} + yx$ is invertible in A, for all $y \in A$.

(d) $\mathbf{1} + xy$ is invertible in A, for all $y \in A$.

Consequently, $\operatorname{Rad} A$ *is a closed two-sided ideal and every $x \in \operatorname{Rad} A$ is quasi-nilpotent (i. e. $\rho(x) = 0$).*

Proof. (a)⇒(c) Let $y \in A$ and suppose that $\mathbf{1} + yx$ is not left-invertible in A. Then $A(1 + yx)$ is a proper left ideal of A, so contained in a maximal left ideal I of A. Hence we have $yx \in I$ (since $x \in I$) and $1 + yx \in I$, so $\mathbf{1} \in I$, which leads to the contradiction $I = A$. Thus $1 + yx$ is left-invertible.

Now choose $z \in A$ with $(\mathbf{1} + z)(\mathbf{1} + yx) = \mathbf{1}$, that is $z + yx + zyx = 0$. As $x \in I$ for all maximal left ideals I of A, the same holds for z. Hence the argument above shows that $\mathbf{1} + z$ is left-invertible. But then $\mathbf{1} + z$ is invertible and therefore so is $\mathbf{1} + yx$.

(c)⇒(a) If there is a maximal left ideal I of A such that $x \notin I$, then $I + Ax = A$, hence $\mathbf{1} - yx \in I$ for some $y \in A$. But $\mathbf{1} - yx$ is invertible, hence $I = A$ which is a contradiction.

(c)⇔(d) If $x, y \in A$, we have that $\mathbf{1} + xy$ is invertible if and only if $\mathbf{1} + yx$ is invertible. In fact, if $z := (1 + xy)^{-1}$, then it is straightforward to check that $1 - yzx$ is the inverse of $1 + yx$.

(a)⇔(b) By the equivalence of (c) and (d), the statement (a) is seen to be left-right symmetric, so that (a) and (b) are equivalent as well. □

Proposition A.1.3 *Let A be a Banach algebra, then we have:*

(a) $\tilde{A} := A/\operatorname{Rad} A$ *is semi-simple.*

(b) The coset $\tilde{x}$ is invertible in $\tilde{A}$ if and only if x is invertible in A. Therefore, $\operatorname{Sp}_A x = \operatorname{Sp}_{\tilde{A}} \tilde{x}$ and $\rho_A(x) = \rho_{\tilde{A}}(\tilde{x})$.

Proof. (a) Let I' be a maximal left ideal of $\tilde{A}$, then $I = \{x : \ \tilde{x} \in I'\}$ is a left ideal of A. It is maximal because if J is a left ideal containing I, then $I' \subseteq \tilde{J} = \{\tilde{x} : \ x \in J\}$, so that $I' = J$ and thus $I = J$. Conversely, if I is a maximal left ideal of A, then $\tilde{I}$ is a maximal left ideal of $\tilde{A}$ for similar reasons. Hence $\{x : \ \tilde{x} \in \operatorname{Rad}(\tilde{A})\}$ is in the intersection of all maximal left ideals of A, that is the radical of A. Hence $\operatorname{Rad}(\tilde{A}) = \{\tilde{0}\}$ and hence $\tilde{A}$ is semi-simple.

(b) If x is invertible, then there exists $y \in A$ such that $yx = xy = 1$. Thus $\tilde{y}\tilde{x} = \tilde{x}\tilde{y} = \tilde{1}$, hence $\tilde{x}$ is invertible. Conversely, if $\tilde{x}$ is invertible, there exists $y \in A$ such that $\tilde{y}\tilde{x} = \tilde{x}\tilde{y} = \tilde{1}$. Then $xy = 1 + u$ and $yx = 1 + v$ with $u, v \in \operatorname{Rad} A$, so by Proposition A.1.2, $1 + u$ and $1 + v$ are invertible in A. Hence $xy(1 + u)^{-1} = 1$ and $(1 + v)^{-1}yx = 1$ and thus x is invertible. □

A.2 Representation theory

Definition A.2.1 *Let A be a Banach algebra and X be a complex vector space $\neq \{0\}$. A homomorphism $\pi : A \to \mathcal{L}(X)$ into the algebra of operators on X is called a* representation *of A on X.*

If a linear subspace $Y \subseteq X$ satisfies $\pi(x)Y \subseteq Y$ for all $x \in A$, we say that Y is invariant under $\pi(x)$. *A representation π is said to be* irreducible *if the only linear subspaces of X invariant under $\pi(x)$ are $\{0\}$ and X.*

A representation π is said to be bounded *if X is a Banach space and if $\pi(x)$ is a bounded linear operator on X for all $x \in A$. Moreover it is said to be* continuous *if it is bounded and if there exists a constant $C > 0$ such that $||\pi(x)|| \leq C||x||$ for all $x \in A$.*

Example A.2.2 *Let I be a maximal left ideal of a Banach algebra A. Let $X = A/I$ be the Banach space with the norm $|||\tilde{a}||| = \inf_{u \in I} \|a + u\|$. Then π defined by $\pi(x)\tilde{a} = \widetilde{xa}$ is an irreducible continuous representation of A on X, called the* left regular representation *associated to I.*

Proof. First note that since every maximal left ideal is closed, A/I with the given norm is indeed a Banach space.

It is obvious that $\pi : A \to \mathcal{L}(X)$ is a homomorphism. To check continuity, if $x \in A$, we have $xI \subseteq I$ and therefore

$$\|\pi(x)\tilde{a}\| = |||\widetilde{xa}||| = \inf_{u \in I} \|xa + u\| \leq \inf_{u \in I} \|x(a+u)\| \leq \|x\| \cdot |||\tilde{a}|||,$$

so that $\|\pi(x)\| \leq \|x\|$.

It remains to check that π is irreducible. Now if $Y \neq \{0\}$ is an invariant linear subspace of X, $J := \{y \in A : \ \tilde{y} \in Y\}$ is a left ideal that properly contains I. By maximality of I we must have $J = A$ and hence $Y = X$ as required. □

Definition A.2.3 *Write $(I : A)$ for the kernel of the representation in the preceding example, so that*

$$(I : A) = \{x \in A : \ xA \subseteq I\}.$$

Proposition A.2.4 *For every irreducible representation π of A there exists a maximal left ideal I such that* $\ker \pi = (I : A)$.

Proof. Let π be an irreducible representation of A on some complex vector space X. Choose $\xi \neq 0$ in X arbitrarily and set

$$I = \{x \in A : \ \pi(x)\xi = 0\}.$$

We prove that I is a maximal left ideal such that $\ker \pi = (I : A)$.

Clearly, I is a left ideal and $I \neq A$. If $J \supset I$, $J \neq I$ is a left ideal, then $Y = \{\pi(x)\xi : \ x \in J\} \neq \{0\}$ is an invariant subspace under $\pi(x)$. Now $Y = X$ by assumption so there exists $e \in J$ such that $\pi(e)\xi = \xi$. Hence for all $x \in A$, we have $xe - x \in I$ and therefore $x = (xe - x) + xe \in I + J = J$ so that $J = A$. This shows that I is maximal.

Now if $x \in \ker \pi$, we have $\pi(xa)\xi = \pi(x)\pi(a)\xi = 0$ for all $a \in A$ so that $xA \subseteq I$ and $x \in (I : A)$. Conversely, if $x \in A$ such that $xA \subseteq I$, we have $\pi(x)\pi(a)\xi = \pi(xa)\xi = 0$ for each $a \in A$. But $F = \{\pi(a)\xi : \ a \in A\} \neq \{0\}$ is an invariant subspace under $\pi(x)$ so that $F = X$ and therefore $\pi(x)X = \{0\}$. Thus $x \in \ker \pi$, so that $\ker \pi = (I : A)$ is proved. □

By the preceding proposition, we have

$$\begin{aligned} \{\ker \pi : \ \pi \text{ irreducible repr.}\} &\subseteq \{(I : A) : \ I \text{ maximal left ideal}\} \\ &\subseteq \{\ker \pi : \ \pi \text{ continuous irreducible repr.}\}, \end{aligned}$$

hence these sets are equal. Furthermore, we have

$$\operatorname{Rad} A = \bigcap \{(I : A) : \ I \text{ maximal left ideal}\}.$$

Indeed, if $x \in \operatorname{Rad} A$ we have $xA \subseteq \operatorname{Rad} A \subseteq I$ for every maximal left ideal I, so that $x \in (I : A)$. On the other hand, if $x \in (I : A)$ for every maximal left ideal I, then we have $x = x \cdot \mathbf{1} \in I$; thus $x \in \operatorname{Rad} A$.

In particular, we have proven

Proposition A.2.5

$$\operatorname{Rad} A = \bigcap \{\ker \pi : \ \pi \text{ a continuous irreducible representation}\}. \ \square$$

Remark: If A is commutative, every character $\chi : A \to \mathbb{C}$ is a continuous irreducible representation on $X = \mathbb{C}$. In fact, these are all irreducible representations.

For, if π is a irreducible representation, then $\ker \pi = (I : A) = I$ for some maximal ideal I. By the theory of commutative Banach algebras there exists a character χ with $\ker \chi = I = \ker \pi$. We have $A/I \cong \mathbb{C}$, hence if $a \in A$, we can write $a = \lambda \mathbf{1} + u$ with $\lambda \in \mathbb{C}$ and $u \in I$. We therefore have

$$\pi(a) = \pi(\lambda \mathbf{1}) = \lambda \mathbf{1} = \chi(\lambda I) = \chi(a),$$

so that $\pi = \chi$ as desired. $\square$

A.3 Some results using representation theory

Some of the following statements are used in chapter 3 but they are using representation theory rather then analytic or subharmonic methods. They will depend on the following important theorem (for a proof see e. g. [4], Theorem 2.1.2., or [2], Theorem 4.2.5.).

Theorem A.3.1 (Jacobson density theorem) *Let π be a continuous irreducible representation of A on a Banach space X. If $\xi_1, \ldots, \xi_n$ are linearly independent in X and if $\eta_1, \ldots, \eta_n$ are in X, then there exists $a \in A$ such that $\pi(a)\xi_i = \eta_i$ for $1 \leq i \leq n$.* $\square$

We will use this theorem to deduce some facts about Banach algebras with finite spectrum.

Lemma A.3.2 *Let A be a Banach algebra, let $n \geq 1$ and suppose $\# \operatorname{Sp} x \leq n$ for $x \in A$. For every continuous irreducible representation π of A on a Banach space X, we have $\dim X \leq n$.*

Proof. Let π be a continuous irreducible representation of A on a Banach space X and suppose $\dim X > n$. Then there exists $\xi_1, \dots, \xi_{n+1}$ linear independent vectors in X. By the Jacobson density theorem, there exists $x \in A$ such that $\pi(x)\xi_i = i\xi_i$, $1 \le i \le n+1$. Hence $\{1, \dots, n+1\} \subseteq \operatorname{Sp} \pi(x) \subseteq \operatorname{Sp} x$ contradiciting $\# \operatorname{Sp} x \le n$. $\square$

For the proof of the next Lemma, see e. g. [2], Lemma 5.4.1.

Lemma A.3.3 *Let A be a semi-simple Banach algebra, let $m \ge 1$ and suppose that every $x \in A$ is algebraic of degree $\le m$. Then A is finite-dimensional.* $\square$

Theorem A.3.4 *Let A be a semi-simple Banach algebra and suppose there exists $n \ge 1$ such that $\# \operatorname{Sp} x \le n$ for $x \in A$. Then A is finite-dimensional.*

Proof. By the preceding Lemma it suffices to show that every $x \in A$ is algebraic of degree $\le n^2$. Now let $x \in A$ and $\operatorname{Sp} x = \{\lambda_1, \dots, \lambda_m\}$, $m \le n$. Let π be a continuous irreducible representation of A on X and suppose $\lambda_1, \dots, \lambda_l \in \operatorname{Sp} \pi(x)$, $l \le m$. Now $\dim X \le n$ by Lemma A.3.2, so by the Caley-Hamilton theorem, we have $(\pi(x) - \lambda_1)^n \cdots (\pi(x) - \lambda_m)^n = 0$. This being true for all such representation π, we have $(x - \lambda_1)^n \cdots (x - \lambda_m)^n = 0$ because A is semi-simple, so that x is algebraic of degree $\le n^2$. $\square$

For the next application we need the following theorem, proved by I. Kaplansky in his book *Inifinite Abelian Groups*, Ann Arbor, 1969. For another proof see [2], Theorem 4.2.7.

Theorem A.3.5 (I. Kaplansky) *Let X be a complex vector space and let T be a linear operator from X into X. Suppose that there exists an integer $n \ge 1$ such that $\xi, T\xi, \dots, T^n\xi$ are linearly dependent for all $\xi \in X$. Then T is algebraic of degree less than or equal to n.* $\square$

This theorem combined with the Jacobson density theorem yields a lemma used in chapter 3.

Lemma A.3.6 *Let A be a Banach algebra and $a \in A$ such that $\# \operatorname{Sp}(ax - xa) = 1$ for all $x \in A$. Then for every continuous irreducible representation π on A we have $\pi(a) = \alpha \mathbf{1}$ for some $\alpha \in \mathbb{C}$.*

Proof. Let π be a continuous irreducible representation of A on a Banach space X. First we show that $\pi(a)$ is algebraic of degree ≤ 2. Suppose the contrary, then by Theorem A.3.5 there exists $\xi \in X$ such that ξ, $\eta = \pi(a)\xi$

and $\pi(a)\eta$ are linearly independent. By the Jacobson density theorem, we then can choose $x \in A$ such that

$$\pi(x)\xi = 0, \qquad \pi(x)\eta = -\xi, \quad \pi(x)\pi(a)\eta = -\eta.$$

Then $\pi(ax - xa)\xi = \pi(a)0 - \pi(x)\eta = \xi$ and $\pi(ax - xa)\eta = \pi(a)(-\xi) + \eta = -\eta + \eta = 0$ so that $\{0, 1\} \subseteq \operatorname{Sp}\pi(ax - xa) \subseteq \operatorname{Sp}(ax - xa)$, contradicting $\#\operatorname{Sp}(ax - xa) = 1$.

Hence, $\pi(a)$ is algebraic of degree ≤ 2 and we can write $\pi(a)^2 = \beta_1\pi(a) + \beta_2\mathbf{1}$ with some $\beta_1, \beta_2 \in \mathbb{C}$. Let $a' := a - \frac{\beta_1}{2}\mathbf{1}$ so that $\pi(a')^2 = (\beta_1 + \frac{\beta_2^2}{4})\mathbf{1} =: \gamma\mathbf{1}$.

Clearly, it suffices to show that $\pi(a') = \alpha'\mathbf{1}$ for some $\alpha' \in \mathbb{C}$. If we again suppose the contrary, there exists $\xi \in X$ such that ξ and $\eta = \pi(a')\xi$ are linearly independent. Hence by the Jacobson density theorem, we can choose $x \in A$ such that

$$\pi(x)\xi = \xi, \qquad \pi(x)\eta = \xi + \eta.$$

Now $\pi(a'x - xa')\xi = \eta - (\xi + \eta) = -\eta$ and since $\pi(a')\eta = \pi(a')^2\xi = \gamma\xi$, we have $\pi(a'x - xa')\eta = \pi(a')(\xi + \eta) - \pi(x)\gamma\xi = \eta + \gamma\xi - \gamma\xi = \eta$, so that $\{-1, 1\} \subseteq \operatorname{Sp}\pi(a'x - xa') \subseteq \operatorname{Sp}(ax - xa)$, a contradiction. Hence the lemma is proved. $\square$

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